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# 半正定分块矩阵和一个线性矩阵方程及其反问题


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## 摘 要

一个实的(未必对称)$n \times n$ 矩阵 $A$ 称为半正定的，如果对任意非零的 $n$ 维行向量 $x$，均有 $xMx^t \geq 0$. 本文给出了一个分块 $n \times n$ 矩阵为半正定的充要条件. 另外，我们讨论了线性矩阵方程 $AX = B$ 对解附加种种条件下的解. 我们应用矩阵在相抵下的标准形给出了这一方程的相容性的充要条件. 还给出这个方程的反问题在对解附加各种条件下的解.

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# Positive Semidefinite Partitioned Matrices and a Linear Matrix Equation and its Inverse Problem *

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**Abstract** A real (may not symmetric) $n\times n$ matrix $M$ is said to be positive semidefinite if, for any real nonzero $n$ dimensional row vector $x, xMx^t \geq 0$. In this paper, we give a necessary and sufficient condition for determining whether a partitioned $n \times n$ matrix is positive semidefinite. Moreover, we consider the solutions of linear matrix equation $AX = B$ with variant conditions on the solutions. The necessary and sufficient conditions for the consistency of this equation are derived using the canonical form of a matrix under the equivalence. The iverse problem of this equation with variant conditions on the solutions is also included.

## I. Introduction

Let $\boldsymbol{R}^{m\times n}$ be the set of all real $m \times n$ matrices, $S^{m\times n}$ the set of all real symmetric $n \times n$ matrices. And let $\boldsymbol{R}^n$ denote the space of $n$ dimensional real row vectors. Recall that an $n \times n$ matrix $M$ is said to be positive semidefinite (positive definite) if, for any nonzero $x \in\boldsymbol{R}^n, xMx^t \geq 0 (> 0)$.

We consider the solutions of the linear matrix equation

$$AX = B, \tag{1}$$

where $A, B \in \boldsymbol{R}^{m\times n}$. Many authors have studied the symmetric solutions of the equation (1). The method of Vetter [1] and Magnus and Neudecker [2] is to use the symmetric condition to reduce the dimension of the vector of unknowns from $n^2$ to $\frac{1}{2}n(n+1)$. Using the partitioned minimum-norm reflexive generalized inverse, Don [3] derived the general symmetric solutions of the equation (1). Dai [4] gave a necessary and sufficient condition for the consistency of the equation (1) with the symmetric solution and the general symmetric solution of the equation (1) using the singular-value decomposition. Moreover, Zhong Guo [5] and Jiong-Sheng Li [6] considered the positive definite solutions of the inverse problem of equation (1) with $X, B \in \boldsymbol{R}^{n\times 1}$. An-Pin Liao [7] discussed the positive semidefinite symmetric solutions of the inverse problem of the equation (1) with $X, B \in \boldsymbol{R}^{m\times n}$.

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The purpose of this paper is to give the solutions of the equation (1) with variant condition on the solutions using the canonical form of a matrix under equivaletce. For this reason, we first derive a necessary and sufficient condition for determining whether a partitioned $n \times n$ matrix is positive semidefinite (Section 2). Next, we study the solutions of the equation (1) with variant condition on the solutions (Section 3). Finally, we consider the inverse problem with various conditions on the matrix $A$ of the equation (1).

## 2. Partitioned Positive Semidefinite Matrices

First we list some simple results without proofs.

**Lemma 2.1** *Let $P \in \mathbf{R}^{n\times n}$ be non- singular. Then $A \in \mathbf{R}^{n\times n}$ is positive semidefinite (or positive definite) if and only if $PAP^t$ is, too. Particularly, $S \in S^{n\times n}$ is positive semidefinite if and only if $PSP^t$ is also.*

**Lemma 2.2** *Every matrix $A \in \mathbf{R}^{n\times n}$ has an unique decomposition into sum of a symmetric matrix $S(A) \in S^{n\times n}$ and a skew symmetric matrix $K(A) \in \mathbf{R}^{n\times n}$, i.e., $A = S(A) + K(A)$, where $S(A) = \frac{1}{2}(A + A^t)$ and $K(A) = \frac{1}{2}(A - A^t)$.*

**Lemma 2.3** *The matrix $A = S(A) + K(A) \in \mathbf{R}^{n\times n}$ is positive semidefinite (or positive definite) if and only if $S(A)$ is positive semidefinite (or positive definite).*

Next, we prove the following

**Theorem 2.4** *Suppose $S_{11} \in S^{s\times s}$. Then $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^t & S_{22} \end{bmatrix} \in S^{n\times n}$ is positive semidefinite if and only if $S_{11}$ is positive semidefinite, and $S_{12} = S_{11}Y$ and $S_{22} = Z + Y^tS_{11}Y$, where $Z \in S^{(n-s)\times(n-s)}$ is positive semidefinite and $Y \in \mathbf{R}^{s\times(n-s)}$.*

**Proof** Assume that $S$ is positive semidefinite. Then $S_{11}$ is also positive semidefinite. Hence, there is a non-singular matrix $P \in \mathbf{R}^{s\times s}$ such that

$$S_{11} = P\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}P^t,$$

where $I_r \in \mathbf{R}^{r\times r}$ is the identity matrix, and $r$ =rank $S_{11}$. By Lemma 2.1, the symmetric matrix

$$\begin{bmatrix} P & 0 \\ 0 & I_{n-s} \end{bmatrix} S \begin{bmatrix} P & 0 \\ 0 & I_{n-s} \end{bmatrix}^t = \begin{bmatrix} PS_{11}P^t & PS_{12} \\ S_{12}^tP^t & S_{22} \end{bmatrix}$$

is positive semidefinite. Denote $S_{12}^tP^t = [\tilde{S}_{12}^t, \tilde{S}_{13}^t], \tilde{S}_{12} \in \mathbf{R}^{r\times(n-s)}, \tilde{S}_{13} \in \mathbf{R}^{(s-r)\times(n-s)}$. Then

$$\begin{bmatrix} P & 0 \\ 0 & I_{n-s} \end{bmatrix} S \begin{bmatrix} P & 0 \\ 0 & I_{n-s} \end{bmatrix}^t = \begin{bmatrix} I_r & 0 & \tilde{S}_{12} \\ 0 & 0 & \tilde{S}_{13} \\ \tilde{S}_{12}^t & \tilde{S}_{13}^t & S_{22} \end{bmatrix}$$

Since the above symmetric matrix is positive semidefinite, we have $\tilde{S}_{13} = 0$. Consequently,

$$PS_{12} = \begin{bmatrix} \tilde{S}_{12} \\ 0 \end{bmatrix} = PS_{11}P^tPS_{12},$$

i.e., $P(S_{12} - S_{11}P^tPS_{12}) = 0$. But $P$ is non-singular. So $S_{12} = S_{11}(P^tPS_{12})$. Put $Y = P^tPS_{12}$. From Lemma 2.1, the symmetric matrix

$$\begin{bmatrix} I_s & -Y \\ 0 & I_{n-s} \end{bmatrix}^t S \begin{bmatrix} I_s & -Y \\ 0 & I_{n-s} \end{bmatrix} = \mathrm{diag}(S_{11}, S_{22} - Y^tS_{11}Y)$$

is positive semidefinite. Thus $Z = S_{22} - Y^tS_{11}Y \in S^{(n-s)\times(n-s)}$ is positive semidefinite.

Conversely, since $S_{11}$ and $Z$ are positive semidefinite, we have by Lemma 2.1 that the matrix

$$\begin{bmatrix} I_s & Y \\ 0 & I_{n-s} \end{bmatrix} \mathrm{diag}(S_{11}, Z) \begin{bmatrix} I_s & Y \\ 0 & I_{n-s} \end{bmatrix} = S$$

is positive semidefinite.

This proves the theorem.

**Corollary 2.5** *For $S_{11} \in S^{s\times s}$, the matrix $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^t & S_{22} \end{bmatrix} \in S^{s\times s}$ is positive definite if and only if $S_{11}$ is positive definite, and there is a positive definite symmetric matrix $Z \in S^{(n-s)\times(n-s)}$ such that $S_{22} = Z + S_{12}^tS_{11}^{-1}S_{12}$.*

**Proof** Assume $S$ is positive definite. Then $S_{11}$ is positive definite and non-singular. Consequently,

$$\begin{bmatrix} I_s & -S_{11}^{-1}S_{12} \\ 0 & I_{n-s} \end{bmatrix}^t S \begin{bmatrix} I_s & -S_{11}^{-1}S_{12} \\ 0 & I_{n-s} \end{bmatrix} = \mathrm{diag}(S_{11}, S_{22} - S_{12}^tS_{11}^{-1}S_{12})$$

is positive definite. Thus, $Z = S_{22} - S_{12}^tS_{11}^{-1}S_{12} \in S^{(n-s)\times(n-s)}$ is positive definite. Conversely, since $S_{11}$ and $Z = S_{22} - S_{12}^tS_{11}^{-1}S_{12}$ are positive definite, the matrix

$$\begin{bmatrix} I_s & S_{11}^{-1}S_{12} \\ 0 & I_{n-s} \end{bmatrix}^t \mathrm{diag}(S_{11}, Z) \begin{bmatrix} I_s & S_{11}^{-1}S_{12} \\ 0 & I_{n-s} \end{bmatrix} = S$$

is positive definite.

Now we give a characterization of the partitioned positive semidefinite matrices.

**Theorem 2.6** *Let $A_{11} \in R^{s\times s}$. Then the partitioned matrix $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21}^t & A_{22} \end{bmatrix} \in R^{n\times n}$ is positive semidefinite if and only if $A_{11}$ is positive semidefinite, and $A_{21} = -A_{12} + (A_{11} + A_{11}^t)Y$ and $A_{22} = Z + Y^tA_{11}Y$, where $Z \in R^{n\times n}$ is positive semidefinite and $Y \in R^{s\times(n-s)}$.*

**Proof** From Lemma 2.3, the matrix $A$ is positive semidefinite if and only if $S(A)$ is, too. Note that

$$2S(A) = \begin{bmatrix} A_{11} + A_{11}^t & A_{12} + A_{21} \\ A_{12}^t + A_{21}^t & A_{22} + A_{22}^t \end{bmatrix}.$$

By Theorem 2.4, the symmetric matrix $2S(A)$ is positive semidefinite if and only if $A_{11} + A_{11}^t$ is also positive semidefinite, and $A_{12} + A_{21} = (A_{11} + A_{11}^t)Y$ and $A_{22} + A_{22}^t = W +$

$Y^t(A_{11}+A_{11}^t)Y$, where $W \in S^{(n-s)\times(n-s)}$ is positive semidefinite and $Y \in \boldsymbol{R}^{s\times(n-s)}$. By Lemma 2.3, the symmetric matrix $A_{11}+A_{11}^t$ is positive semidefinite if and only if $A_{11}$ is, too, and the symmetric matrix $W = A_{22}+A_{22}^t - Y^t(A_{11}+A_{11}^t)Y$ is positive semidefinite if and only if $A_{22} - Y^tA_{11}Y = Z \in \boldsymbol{R}^{(n-s)\times(n-s)}$ is positive semidefinite.

The proof of Theorem 2.6 is completed.

**Corollary 2.7** *Suppose $A_{11} \in \boldsymbol{R}^{s\times s}$. Then the matrix $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21}^t & A_{22} \end{bmatrix} \in \boldsymbol{R}^{n\times n}$ is positive difinite if and only if $A_{11}$ is positive definite and there exists o positive definite matrix $Z \in \boldsymbol{R}^{(n-s)\times(n-s)}$ such that*

$$A_{22} = Z + (A_{12}+A_{21})^t(A_{11}+A_{11}^t)^{-1}A_{11}(A_{11}+A_{11}^t)^{-1}(A_{12}+A_{21}).$$

**Proof** It is enough to take $Y = (A_{12}+A_{21})(A_{11}+A_{11}^t)^{-1}$ in Theorem 2.6.

## 3. Solutions of the Equation (1)

First, we consider the solutions of the equation (1) with $A, B \in \boldsymbol{R}^{m\times n}$. It is well-known that, for $A \in \boldsymbol{R}^{m\times n}$, there are non-singular matrices $P \in \boldsymbol{R}^{m\times n}$ such that

$$A = P\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}Q, \tag{2}$$

where $r$ =rank $A$. The matrix in the right-hand side of (2) is said to be the canonical form of $A$ under equivalence. Clearly, the pair $(P,Q)$ in (2) is fixed, not unique. To study the solutions of the equation (1), we assume that the pair $(P,Q)$ in (2) is fixed, and the matrix $B \in \boldsymbol{R}^{m\times n}$ has the following form:

$$B = P\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}(Q^{-1})^t, \tag{3}$$

where $B_{11} \in \boldsymbol{R}^{r\times r}$.

Now we prove the following

**Theorem 3.1** *Suppose that A and B have the forms (2) and (3) respectively. Then the equation (1) has a solution if and only if $B_{21} = 0$ and $B_{22} = 0$. In that case it has the general solution*

$$X = Q^{-1}\begin{bmatrix} B_{11} & B_{12} \\ X_{21} & X_{22} \end{bmatrix}(Q^{-1})^t, \tag{4}$$

*where $X_{21} \in \boldsymbol{R}^{(n-r)\times r}$ and $X_{22} \in \boldsymbol{R}^{(n-r)\times(n-r)}$ are arbitrary.*

**Proof** Suppose

$$X = Q^{-1}\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}(Q^{-1})^t$$

is a solution of the equation (1), where $X_{11} \in \boldsymbol{R}^{r\times r}$. Then,

$$\begin{aligned} P\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}^t (Q^{-1})^t &= B = AX = P\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} QQ^{-1}\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}(Q^{-1})^t \\ &= P\begin{bmatrix} X_{11} & X_{12} \\ 0 & 0 \end{bmatrix}(Q^{-1})^t, \end{aligned}$$

i.e.,

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} \\ 0 & 0 \end{bmatrix}$$

Thus, $X_{11} = B_{11}, X_{12} = B_{12}, B_{21} = 0$, and $B_{22} = 0$. This shows that $X$ has form (4).

Conversely, suppose $B_{21} = 0$, and $B_{22} = 0$. Take a matrix $X$ of type (4). It is easy to verify that the matrix $X$ of type (4) is a solution of the equation (1).

This proves the theorem.

To discuss further the solutions of the equation (1) with various conditions on the solutions, we need the following lemma.

**Lemma 3.2** *Suppose $P = [P_1, P_2], (P^{-1})^t = [R_1^t, R_2^t], Q^t = [Q_1^t, Q_2^t]$ and $Q^{-1} = [T_1, T_2]$, where $P_1, R_1^t \in \boldsymbol{R}^{m\times r}, P_2, R_2^t \in \boldsymbol{R}^{m\times(m-r)}, Q_1^t, T_1 \in \boldsymbol{R}^{n\times r}$ and $Q_2^t, T_2 \in \boldsymbol{R}^{n\times(n-r)}$. Then,*

$$\begin{aligned} P_1R_1 + P_2R_2 &= I_m, & (5)\\ R_1P_1 &= I_r, R_1P_2 = 0, R_2P_1 = 0, R_2P_2 = I_{m-r}, & (6)\\ Q_1T_1 &= I_r, Q_1T_2 = 0, Q_2T_1 = 0, Q_2T_2 = I_{n-r}, & (7)\\ T_1Q_1 + T_2Q_2 &= I_n, & (8) \end{aligned}$$

*and*

$$\begin{aligned} A &= P_1Q_1, & (9)\\ B_{11} &= R_1BQ_1^t = R_1BA^tR_1^t, B_{12} = R_1BQ_1^t, & \\ B_{21} &= R_2BQ_1^t, \qquad B_{22} = R_2BQ_2^t. & (10) \end{aligned}$$

**Proof** It is easy to see that (5) and (6) are the consequences of $PP^{-1} = I_m$ and $P^{-1}P = I_m$ respectively, (7) and (8) are the corollaries of $QQ^{-1} = I_n$ and $Q^{-1}Q = I_n$ respectively. From (2), we have (9). By (3), we obtain

$$B = P_1B_{11}T_1^t + P_2B_{21}T_1^t + P_1B_{12}T_2^t + P_2B_{22}T_2^t.$$

Consequently, by (6) and (7), $R_iBQ_j^t = B_{ij}, 1 \le i,j \le 2$. Since $A^tR_1^t = Q_1^tP_1^tR_1^t = Q_1^t$, we obtain $B_{11} = R_1BQ_1^t = R_1BA^tR_1^t$.

**Theorem 3.3** *The equation (1) has a solution if and only if $R_2B = 0$. In that case it has the general solution*

$$X = T_1R_1BA^tR_1^tT_1^t + T_1R_1BQ_2^tT_2^t + T_2X_{21}T_1^t + T_2X_{22}T_2^t, \tag{11}$$

*where* $X_{21} \in \boldsymbol{R}^{(n-r)\times r}$ *and* $X_{22} \in \boldsymbol{R}^{(n-r)\times(n-r)}$ *are arbitrary.*

**Proof** From (10), we have $R_2 B = [B_{21}, B_{22}](Q^{-1})^t$. Thus, by Theorem 3.1, the equation (1) has a solution if and only if $R_2 B = 0$. From (4) and Lemma 3.2, the general solution of (1) is

$$\begin{aligned} X &= T_1 B_{11} T_1^t + T_1 B_{12} T_2^t + T_2 X_{21} T_1^t + T_2 X_{22} T_2^t \\ &= T_1 R_1 B A^t R_1^t T_1^t + T_1 R_1 B Q_2^t T_2^t + T_2 X_{21} T_1^t + T_2 X_{22} T_2^t. \end{aligned}$$

This proves the theorem.

Next, we turn to the symmetric solutions of the equation (1) with $A, B \in \boldsymbol{R}^{m\times n}$.

**Theorem 3.4** *The equation (1) has a symmetric solution if and only if* $R_2 B = 0$ *and* $BA^t$ *is symmetric. In that case it has the general solution*

$$X = (Q^{-1}) \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^t & X_{22} \end{bmatrix} (Q^{-1})^t, \tag{12}$$

*or*

$$X = T_1 R_1 B A^t R_1^t T_1^t + T_1 R_1 B Q_2^t T_2^t + T_2 Q_2 B^t R_1^t T_1^t + T_2 X_{22} T_2^t, \tag{13}$$

*where* $X_{22} \in S^{(n-r)\times(n-r)}$ *is arbitrary.*

**Proof** Suppose $X$ is a symmetric solution of (1). Then, by Theorem 3.3, $R_2 B = 0$. Since $X$ is symmetric, the matrix $BA^t = AXA^t$ is, too. In addition, by Theorem 3.1, $X$ has the form (4). But $X$ is symmetric, so $X_{21} = B_{12}^t$ and the matrix $X_{22}$ is also symmetric. Consequently, $X$ has the form (12). Finally, by Theorem 3.3, $X$ has the form (11). Hence,

$$X^t = T_1 R_1 A B^t R_1^t T_1^t + T_2 Q_2 B^t R_1^t T_1^t + T_1 X_{21}^t T_2^t + T_2 X_{22}^t T_2^t.$$

Because of $BA^t = AB^t$ and $X^t = X$, we have

$$T_1(R_1 B Q_2^t - X_{21}^t) T_2^t - T_2(Q_2 B^t R_1^t - X_{21}) T_1^t + T_2(X_{22} - X_{22}^t) T_2^t = 0. \tag{i}$$

Using (7) and (10) on the equality $Q_2 \times (i) \times Q_1^t$, we obtain $X_{21} = Q_2 B^t R_1$. Moreover, it follows from (7) and $Q_2 \times (i) \times Q_2^t$ that $X_{22}^t = X_{22}$. Thus, $X$ has the form (13).

Conversely, because of $R_2 B = 0$, the equation (1) has a solution by Theorem 3.3, and its general solution $X$ has the form (4) or (11). Take $X_{21} = Q_2 B^t R_1^t$, and let $X_{22}$ be a symmetric $(n-r)\times(n-r)$ matrix. Since $BA^t$ is symmetric, the solution $X$ is, too.

This proves the theorem.

Finally, we discuss the positive semidefinite solutions of the equation (1).

**Theorem 3.5** *The equation (1) has a positive semidefinite (not necessarily symmetric) solution if and only if* $R_2 B = 0$ *and the matrix* $BA^t$ *is positive semidefinite. In that case it has the general positive semidefinite solution*

$$X = Q^{-1} \begin{bmatrix} B_{11} & B_{12} \\ -B_{12}^t + Y^t(B_{11} + B_{11}^t) & Z + Y^t B_{11} Y \end{bmatrix} (Q^{-1})^t, \tag{14}$$

*or*

$$\begin{aligned} X &= (T_1+T_2Y^t)R_1BA^tR_1^t(T_1+T_2Y^t)^t+T_2ZT_2^t \\ &\quad +(T_1R_1BQ_2^tT_2^t-T_2Q_2B^tR_1^tT_1^t) \\ &\quad +(T_2Y^tR_1AB^tR_1^tT_1^t-T_1R_1BA^tR_1^tYT_2^t), \end{aligned} \tag{15}$$

*where $Z$ is an arbitrary positive semidefinite $(n-r)\times(n-r)$ matrix and $Y$ is an arbitrary $r\times(n-r)$ matrix.*

**Proof** Suppose $X$ is a positive semidefinite solution of the equation (1). Then, by Theorem 3.3, $R_2B=0$. Moreover, since the solution $X$ is positive semidefinite, the matrix $BA^t=AXA^t$ is, too. From Theorem 3.1, the solution $X$ has the form (4). By, Lemma 2.1, the matrix $\begin{bmatrix} B_{11} & B_{12} \\ X_{21} & X_{22} \end{bmatrix}$ is positive semidefinite. Hence, we obtain from Theorem 2.6 that $B_{11}$ is also positive semidefinite, and $X_{21}^t=-B_{12}+(B_{11}+B_{11}^t)Y, X_{22}=Z+Y^tB_{11}Y$, where $Z\in \boldsymbol{R}^{(n-r)\times(n-r)}$ is positive semidefinite and $Y\in\boldsymbol{R}^{r\times(n-r)}$. Thus, the solution $X$ has the form (14). In addition, by Theorem 3.3, we have

$$\begin{aligned} X &= T_1R_1BA^tR_1^tT_1^t+T_1R_1BQ_2^tT_2^t+T_2X_{21}T_1^t+T_2X_{22}T_2^t \\ &= T_1R_1BA^tR_1^tT_1^t+T_1R_1BQ_2^tT_2^t \\ &\quad +T_2(-B_{12}^t+Y^t(B_{11}+B_{11}^t))T_1^t+T_2(Z+Y^tB_{11}Y)T_2^t. \end{aligned}$$

Form Lemma 3.2, $B_{11}=R_1BA^tR_1, B_{12}=R_1BQ_2^t$. Consequently, we have

$$\begin{aligned} X &= T_1R_1BA^tR_1^tT_1^t+T_2Y^tR_1BA^tR_1^tT_1^t+T_2Y^tR_1AB^tR_1^tT_1^t \\ &\quad +T_2Y^tR_1BA^tR_1^tT_2^t+T_1R_1BQ_2^tT_2^t-T_2Q_2BR_1^tT_1^t+T_2ZT_2^t \\ &= (T_1R_1+T_2Y^tR_1)BA^tR_1^tT_1^t+(T_1R_1+T_2Y^tR_1)BA^tR_1^tYT_2^t \\ &\quad -T_1R_1BA^tR_1^tYT_2^t+T_2Y^tR_1AB^tR_1^tT_1^t \\ &\quad +(T_1R_1BQ_2^tT_2^t-T_2Q_2BR_1^tT_1^t)+T_2ZT_2^t. \end{aligned}$$

Thus, the solution $X$ has the form (15).

Conversely, suppose $R_2B=0$ and $BA^t$ is positive semidefinite. Then, by Lemma 3.2, and Lemma 2.1, $B_{11}=R_1BA^tR_1^t$ is positive semidefinite. From Theorem 2.6, the matrix $X$ of type (14) is positive semidefinite. It is easy to verify that the matrix $X$ of type (14) is a solution of the equation (1). In addition, for the matrix $X$ of type (15), we have

$$S(X)=(T_1+T_2Y^t)R_1BA^tR_1(T_1+T_2Y^t)^t+T_2ZT_2^t.$$

Since $BA^t$ and $Z$ are positive semidefinite, the symmetric matrix $S(A)$ is also positive semidefinite. Hence, the matrix $X$ of type (15) is, too. It is easy to verify that the matrix $X$ of type (15) is a solution of the equation (1).

The proof of Theorem 3.5 is completed.

**Corollary 3.6** *The equation (1) has a positive definite (may not symmetric) solution if and only if $R_2B=0$ and the matrix $BA^t$ is positive definite. In that case the general* *positive definite solution $X$ of (1) has the form (14) or (15) in which $Z \in \boldsymbol{R}^{(n-r)\times(n-r)}$ is positive definite.*

**Proof** It is a direct consequence of Theorem 3.5.

**Corollary 3.7** *The equation (1) has a positive semidefinite symmetric solution if and only if $R_2B = 0$ and the matrix $BA^t$ is positive semidefinite symmetric. In that case its general positive semidefinite symmetric solution has the form (12) or (13) in which $X_{22} = Z + Y^tB_{11}Y$ or $X_{22} = Z + Y^tR_1BA^tR_1^t$, $Z$ is an arbitrary positive semidefinite symmetric $(n-r)\times(n-r)$ matrix, and $Y$ is an arbitrary solution of the matrix equation $B_{12} = B_{11}Y$.*

**Proof** Combining Theorem 3.4, 3.5 and 2.4, we obtain directly the result.

**Corollary 3.8** *The equation (1) has a positive definite symmetric solution if and only if $R_2B = 0$ and the matrix $BA^t$ is positive definite symmetric. In that case its general positive definite symmetric solution $X$ has the form (12) in which $X_{22} = Z + B_{12}^tB_{11}^{-1}B_{12}$, $Z$ is an arbitrary positive definite symmetric $(n-r)\times(n-r)$ matrix.*

**Proof** Obvious.

## 4. Inverse Problem of Equation (1)

In this section, we consider the solutions of various kinds of the inverse problem of equation (1) with $X, B \in \boldsymbol{R}^{n\times m}$ It is well known that, for $X \in \boldsymbol{R}^{n\times m}$, there are nonsingular matrices $P \in \boldsymbol{R}^{m\times m}$ and $Q \in \boldsymbol{R}^{n\times n}$ such that

$$X = Q^t \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} P^t, \tag{16}$$

where $r$ =rank$X$. Moreover, we may denote

$$B = Q^{-1} \begin{bmatrix} B_{11}^t & B_{21}^t \\ B_{12}^t & B_{22}^t \end{bmatrix} P^t, \tag{17}$$

where $B_{11} \in \boldsymbol{R}^{r\times r}$.

Clearly, the inverse problem of the equation (1) with $X, B \in \boldsymbol{R}^{n\times m}$ and $A \in \boldsymbol{R}^{n\times n}$ is equivalent to solue the following equation

$$X^tA^t = B^t, \tag{18}$$

where $A^t$ is unknown. Consequently, for every theorem in Section 3 concerning the solutions of the equation (1) with various conditions on the solutions, there is a duality theorem on the inverse problem of the equation (1). We list the relevant results without proofs as follows.

**Theorem 4.1** *The inverse problem of the equation (1) has a solution if and only if $B_{21} = 0$ and $B_{22} = 0$. In that case it has the general solution*

$$A = Q^{-1} \begin{bmatrix} B_{11}^t & X_{21} \\ B_{12}^t & X_{22} \end{bmatrix} (Q^{-1})^t, \tag{19}$$

*where* $X_{21} \in R^{r\times(n-r)}$ *and* $X_{22} \in R^{(n-r)\times(n-r)}$ *are arbitrary.*

From now on, we suppose that the senses of $P_1, P_2, R_1, R_2, Q_1, Q_2, T_1$ and $T_2$ are the same with Lemma 3.2. Then, we have

**Theorem 4.2** *The inverse problem of the equation (1) has a solution* $A \in R^{n\times n}$ *if and only if* $R_2 B^t = 0$, *in which case it has the general solution*

$$A = T_1 R_1 X^t B R_1^t T_1^t + T_2 Q_2 B R_1^t T_1^t + T_1 X_{21} T_2^t + T_2 X_{22} T_2^t, \tag{20}$$

*where* $X_{21} \in R^{r\times(n-r)}$ *and* $X_{22} \in R^{(n-r)\times(n-r)}$ *are arbitrary.*

**Theorem 4.3** *The inverse problem of the equation (1) has a symmetric solution if and only if* $R_2 B = 0$ *and the matrix* $X^t B$ *is symmetric. In that case it has the general symmetric solution*

$$A = Q^{-1}\begin{bmatrix} B_{11}^t & B_{12} \\ B_{12}^t & X_{22} \end{bmatrix}(Q^{-1})^t, \tag{21}$$

*or*

$$A = T_1 R_1 X^t B R_1^t T_1^t + T_2 Q_2 B R_1^t T_1^t + T_1 R_1 B^t Q_2^t T_2^t + T_2 X_{22} T_2^t, \tag{22}$$

*where* $X_{22}$ *is an arbitrary* $(n-r)\times(n-r)$ *symmetric matrix.*

**Theorem 4.4** *The inverse problem of the equation (1) has a positive semidefinite solution if and only if* $R_2 B^t = 0$ *and the matrix* $X^t B$ *is positive semidefinite, in which case its general positive semidefinite solution is the following*

$$A = Q^{-1}\begin{bmatrix} B_{11} & -B_{12} + (B_{11} + B_{11}^t)Y \\ B_{12}^t & Z + Y^t B_{11} Y \end{bmatrix}(Q^{-1})^t, \tag{23}$$

*or*

$$\begin{aligned} A \;=\;& (T_1 + T_2 Y^t) R_1 X^t B R_1^t (T_1 + T_2 Y^t)^t + T_2 Z T_2^t \\ &+ (T_2 Q_2 B R_1^t T_1^t - T_1 R_1 B Q_2^t T_2^t) \\ &+ (T_1 R_1 B^t X R_1^t Y T_2^t - T_2 Y^t R_1 X^t B R_1^t T_1^t), \end{aligned} \tag{24}$$

*where* $Z$ *is an arbitrary* $(n-r)\times(n-r)$ *positive semidefinite matrix and* $Y$ *is an arbitrary* $r\times(n-r)$ *matrix.*

**Theorem 4.5** *The inverse problem of the equation (1) has a positive definite solution if and only if* $R_2 B^t = 0$ *and the matrix* $X^t B$ *is positive definite, in which case its general positive definite solution* $A$ *has the form (23) or (24), where* $X_{22} = Z + Y^t B_{11}^t Y$ *and* $Z$ *is an arbitrary* $(n-r)\times(n-r)$ *positive definite matrix and* $Y$ *is an atbitrary* $r\times(n-r)$ *matrix.*

**Theorem 4.6** *The inverse problem of the equation (1) has a positive semidefinite symmetric solution if and only if* $R_2 B^t = 0$ *and the matrix* $X^t B$ *is positive semidefinite symmetric. In that case its general positive semidefinite symmetric solution has the form (21) with* $X_{22} = Z + Y^t B_{11} Y$, *or (24), where* $Z$ *is an arbitrary* $(n-r)\times(n-r)$ *positive semidefinite symmetric matrix, and* $Y$ *is an arbitrary* $r\times(n-r)$ *matrix.*

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# 半正定分块矩阵和一个线性矩阵方程及其反问题


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## 摘　要

一个实的(未必对称)$n\times n$ 矩阵 $A$ 称为半正定的，如果对任意非零的 $n$ 维行向量 $x$，均有 $xMx^t\geq 0$. 本文给出了一个分块 $n\times n$ 矩阵为半正定的充要条件. 另外，我们讨论了线性矩阵方程 $AX=B$ 对解附加种种条件下的解. 我们应用矩阵在相抵下的标准形给出了这一方程的相容性的充要条件. 还给出这个方程的反问题在对解附加各种条件下的解.