[3] R. Yue Chi Ming, *Annihilators and strongly regular rings*, Rendiconti Seminario Facolta Scienze, Universita Cagliari, Vol. 57 Fasc., **1**(1987), 51–59.

[4] R. Yue Chi Ming, *On Von Neumann regular rings XV*, ACTA, Mathematica Vietnamica, **13:2**(1988), 71–79.

[5] R. Yue Chi Ming, *On Von Neumann regular rings V*, Math. J. Okayama Univ., **22**(1980), 151–160.

[6] Zhang Jule, *p−injective rings and Von Neumann regular rings*, Northeastern Math. J., **7:3**(1991), 326–331.

[7] V.S. Ramamurthi, *Weakly regular rings*, Canad. Math. Bull., **16:3**(1973), 317–321.

[8] K.R. Goodearl, *Von Neumann Regular Rings*, Pitmen, London, 1979.

关于半交换环与强正则环

杜先能
(安徽师范大学，芜湖241000)

摘　要

本文得到了环R是强正则环的若干充分必要条件，证明了下面条件是等价的:

(1)R是强正则的;
(2)R是半交换正则的;
(3)R是半交换的左SF-环;
(4)R是半交换的ELT环，且使得每个单左R-模是P-内射的或者平坦的;
(5)R是半交换右非奇异的左SF-环;
(6)R是半素的半交换左(或右)P-内射环.

Journal of Mathematical Research & Exposition
Vol.14, No.1, Feb. 1994

On Semicommutative Rings and Strongly Regular Rings*

Du Xianneng
(Dept. of Math., Anhui Normal Univ., Wuhu)

Abstract. This note contains some necessary and sufficient conditions for rings to be strongly regular.

Von Neumann regular rings, strongly regular rings and their generalizations have been studied by many authors, see [1,3,6,8]. In this note, we consider semicommutative rings whose simple left $R-$mudules are either $p-$injective or flat. The following conditions are proved to be equivalent: (1) R is a strongly regular ring. (2) R is a semicommutative and regular ring. (3) R is a semicommutative left SF-ring. (4) R is a semicommutative ELT ring whose simple left $R-$modules are either $p-$injective or flat. (5) R is a semicommutative right nonsingular left $p-$injective ring. (6) R is a semiprime semicommutative left (or right) $p-$injective ring.

Throuthout, R represents an associative ring with identity and $R-$modules are unital. A ring R is said to be semicommutative[1] if $xy = 0$ implies $xRy = 0$ $(x, y \in R)$. It is clear that every reduced ring is semicommutative and the class of semicommutative rings contains left duo rings and right duo rings.

We begin with a characterization of strongly regular rings.

Recall that a ring R is called strongly regular if for each $a \in R$ there exists $b \in R$ such that $a = a^2b$. In [2, Chap.1, §12], a well-known characterization of strongly regular rings was given: R is strongly regular if and only if R is reduced and regular, and if and only if R is regular and duo. Now we have

Theorem 1 *The following conditions are equivalent.*
(1) R is a strongly regular ring.
(2) R is a semicommutative and regular ring.

Proof (1) $\Rightarrow$ (2). R is reduced and regular and hence R is semicommutative and regular.

(2) $\Rightarrow$ (1). For any $x \in R$, there exists $y \in R$ such that $xyx = x$, this gives $(x(1 - xy))^2 = 0$ and hence $x(1 - xy)Rx(1 - xy) = 0$. Since $x(1 - xy) = x(1 - xy)zx(1 - xy)$ for some $z \in R$, then $x(1 - xy) = 0$, i.e., $x = x^2y$. $\square$

A ring R is called a left (right) SF-ring if each simple left (right) $R-$module is flat. It is clear that regular rings are left and right SF-rings, but the problem that whether a left (right) SF-ring is regular is still open. Many authors proved that the regularity of SF-rings requires certain additional conditions. For example, right (left) duo left (right)

*Received Feb. 7, 1992. Research supported by Anhui Provincial Education Council.

SF-rings are regular. Replacing left (right) duo by semicommutative, we have

Theorem 2 *The following conditions are equivalent.*

(1) R is a strongly regular ring.

(2) R is a semicommutative left SF-ring.

(3) R is a left SF-ring whose maximal left ideals are ideals of R.

Proof Obviously, (1) implies (2).

Assume (2). Since R is semicommutative, then $r(a)$, the right annihilator of a, is an ideal of R for any $a \in R$. We consider left ideal $Ra + r(a)$. If M is a maximal left ideal of R containing $Ra + r(a)$, then $a = ab$ for some $b \in R$ since R is left SF-ring. This implies $1 \in M$, a contradiction. Thus $Ra + r(a) = R$. This proves that R is regular and hence R is strongly regular by Theorem 1. Thus every maximal one-sided ideal of R is an ideal and hence (2) implies (3).

Furthermore, it follows from [3, Th.4] that (3) implies (1). □

A ring R is left (right) weakly regular[4] if $L^2 = L$ for each left (right) ideal L of R. V.S. Ramamurthi [4] proved that a reduced left weakly regular ring is right weakly regular. The next theorem extends this result.

Theorem 3 *If R is a semicommutative ring, then R is left weakly regular if and only if R is right weakly regular.*

Proof Suppose R is left weakly regular, then for any $a \in R, a = ba$ for some $b \in RaR$. This gives $(a(1-b))^2 = 0$. Since R is semicommutative, then $l(a(1-b))$, the left annihilator of $a(1-b)$, is an ideal of R and

$$a(1-b) \in Ra(1-b) \cap l(a(1-b)).$$

By [4, Remark 3],

$$Ra(1-b) \cap l(a(1-b)) = l(a(1-b))Ra(1-b) = 0,$$

thus $a = ab \in (aR)^2$ and hence R is right weakly regular by [7, Prop. 1].

The converse can be similarly proved. □

A left R-module M is p-injective [4] in case each R-homomorphism from a principal left ideal of R to M can be extended to one from R to M. R is called left p-injective if ${}_RR$ is left p-injective.

Theorem 4 *The following conditions are equivalent.*

(1) R is strongly regular ring.

(2) Every simple left R-module is either p-injective or flat and R is semicommutative ring whose essential left ideals are ideals.

Proof It is obvious that (1) implies (2).

Assume (2). Since R is semicommutative, then $r(a)$ ia an ideal of R for any $a \in R$. We show $Ra + r(a) = R$ for any $a \in R$. Suppose $L = Ra + r(a) \neq R$ for some $a \in R$. Let M be a maximal left ideal of R containing L. Then M is either essential or a summand. If M is a summand, then $M = Re$ for some idempotent $e \neq 1$. Thus $1 - e \in r(M) \subseteq r(A) \subseteq M$

and hence $1 \in M$, a contradiction. This proves that M is essential left ideal of R and hence M is an ideal of R.

If R/M is $p-$injective, take $f: Ra \to R/M$ defined by $xa \to x+M$, then there exists $b \in R$ such that $1+M = ab+M$. This implies $1 \in M$ since M is an ideal of R, a contradiction.

If R/M is flat, then there exists $d \in M$ such that $a = ad$. Thus $1-d \in r(a) \subseteq M$ and $1 \in M$ which is a contradiction. Therefore $R = Ra + r(a)$ for each $a \in R$. This implies that R is regular. By Theorem 1, R is strongly regular and hence (2) implies (1). □

For $p-$injective rings, we have

Proposition 5 *If R is a semicommutative left $p-$injective ring, then R is right duo.*

Proof Suppose K is a right ideal of R. Since R is left $p-$injective, then $r(l(a)) = aR$ for each $a \in R$. Since R is semicommutative, then $l(a)$ is an ideal of R and hence $aR = r(l(a))$ is an ideal of R. This implies $Ra \subseteq aR \subseteq K$ which proves that K is an ideal of R. □

We use Z to denote the right singular ideal of R. The following proposition gives a sufficient condition for R/Z to be reduced.

Proposition 6 *If R is semicommutative left $p-$injective, then R/Z is reduced.*

Proof Let $a \in R, a \notin Z$ such that $a^2 \in Z$, then $r(a)$ is not essential in $r(a^2)$. Let K be a right ideal such that $r(a) \oplus K$ is essential in $r(a^2)$. Let C be a relative complement of $r(a)$ in R such that $K \subseteq C$, then K and C are ideals of R by Proposition 5. Thus $aK \subseteq aC \subseteq C$ which gives $aK \subseteq C \cap r(a) = 0$. Thus $K = K \cap r(a) = 0$ which contradicts $k \neq 0$. This proves that R/Z is reduced. □

Corollary 7 *If R is a semicommutative right nonsingular left $p-$injective ring, then R is strongly regular.*

Proof By Proposition 6, R is reduced. It follows from [6, Th.6] that R is strongly regular. □

Call R an ELT ring [5] if every essential left ideal is an ideal of R. In [5], R. Yue Chi Ming proposed the following question: Is R a von Neumann regular if R is a semiprime ELT left $p-$injective ring ? Replacing ELT by semicommutative, we have

Theorem 8 *If R is a semiprime semicommutative left (or right) $p-$injective ring, then R is strongly regular.*

Proof Assume R is left $p-$injective(The proof in the right $p-$injective case follows symmetrically). By Proposition 5, every right ideal of R is an ideal of R. It follows from [1, Th.1.3] and Theorem 1 that R is strongly regular. □

References

[1] Y. Hirano and H. Tominaga, *Regular rings, V−rings and their generalizations*, Hiroshima Math. J., **9**(1979), 137–149.

[2] B. Stenstrom, *Rings of Quotients*, Springer-Verlag, Berlin-Heidelberg-New York, 1975.

[3] R. Yue Chi Ming, *Annihilators and strongly regular rings*, Rendiconti Seminario Facolta Scienze, Universita Cagliari, Vol. 57 Fasc., **1**(1987), 51–59.

[4] R. Yue Chi Ming, *On Von Neumann regular rings XV*, ACTA, Mathematica Vietnamica, **13:2**(1988), 71–79.

[5] R. Yue Chi Ming, *On Von Neumann regular rings V*, Math. J. Okayama Univ., **22**(1980), 151–160.

[6] Zhang Jule, *p−injective rings and Von Neumann regular rings*, Northeastern Math. J., **7:3**(1991), 326–331.

[7] V.S. Ramamurthi, *Weakly regular rings*, Canad. Math. Bull., **16:3**(1973), 317–321.

[8] K.R. Goodearl, *Von Neumann Regular Rings*, Pitmen, London, 1979.

关于半交换环与强正则环

杜先能
(安徽师范大学，芜湖241000)

摘　　要

本文得到了环R是强正则环的若干充分必要条件，证明了下面条件是等价的:

(1)R是强正则的;
(2)R是半交换正则的;
(3)R是半交换的左SF-环;
(4)R是半交换的ELT环，且使得每个单左R-模是P-内射的或者平坦的;
(5)R是半交换右非奇异的左SF-环;
(6)R是半素的半交换左(或右)P-内射环.