References

[1] Y.Alavi, A. J.Boals, G.Chartrand, P. Erdös, and O. R.Oellermann, *The ascending subgraph decomposition problem,* Congr. Numer. **58**(1987)1–7.

[2] V. G.Vining, *On the estimate of p-graph,* Diskret Analiz., **3**(1964), 25–30.

关于图升分解为独立边集问题

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摘　要

Alavi [1] 给出了图的升分解概念，并猜想每一图都可升分解．本文证明了边数为 $\binom{n+1}{2}$ 的图 G 当边色数 $X'(G) \le (n+2)/2$ 时可升分解为 $\{G_i\}, 1 \le i \le n, G_i \cong iK_2$.

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On the Problem of Ascending Subgraph Decompositions into Matchings *

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Abstract Alavi has given the definition of the ascending subgraph decomposition. He conjectured that every graph of positive size has an ascending subgraph decomposition. In this paper it is proved that a graph G of size $\binom{n+1}{2}$ has an ascending subgraph decomposition $\{G_i\}, 1 \le i \le n$, with $G_i \cong iK_2$, if the edge chromatic number $x'(G) \le (n+2)/2$.

I. Introductions

Yousef Alavi and others have given the definition of the ascending subgraph decompo -sition[1]. Let G be a graph of potive size q, and n be a positive integer for which

$$\binom{n+1}{2} \le q < \binom{n+2}{2}.$$

Then G is said to have an ascending subgraph decomposition if G can be decoposed into n subgraphs $G_1, G_2, \cdots, G_n$ without isolated vertices such that G_i is isomorphic to a proper subgraph of G_{i+1} for $1 \le i \le n-1$. Several classes of graphs possessing an ascending subgraph decomposition are described in [1]. Here we have further results. We will use the following notations.

$|G|$—size of a graph G,
$\Delta(G)$—maximum degree of a graph G,
$x'(G)$—edge chromatic number of a graph G,
$G_1 \cap G_2$—the cap of graph G_1 and G_2,
$G_1 \cup G_2$—the union of graph G_1 and G_2,
$G_1 - G_2$—the difference of graph G_1 and G_2.

A matching of a graph G is a subgraph of G without isolated vertices in which no two edges are adjacent.

II. Some Lemmas

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Lemma 1 *Let E_1 and E_2 be two edge-disjoint matchings of a graph G. If $|E_1| > 2|E_2|$, then there exists a subgraph H of E_1 such that $|H| \geq |E_1| - 2|E_2|$ and $H \cap E_2 = \emptyset$ (i.e., H and E_2 are both vertex-disjoint and edge-disjoint).*

Lemma 2 *Let E_1 and E_2 be two edge-disjoint matchings of a graph G. If $|E_2| \leq n-2$ and $|E_1| \geq 2n-1$, then there exists $H \subset E_1$ such that $G_n = E_2 \cup H \cong nK_2$.*

Lemma 3 *Let E_1 and E_2 be two edge-disjoint matchings of a graph G. If $\frac{n}{2} \leq |E_2| \leq n-2$ and $\frac{3}{2}n \leq |E_1| \leq 2n-2$, then there exists $H_1 \subset E_1$ and $H_2 \subset E_2$ such that*

$$\begin{aligned} G_n &= H_1 \bigcup H_2 \cong nK_2, \\ G_{n-1} &= E_1 - H_1 \cong (n-1)K_2. \end{aligned}$$

Lemma 4 *Let E_1 and E_2 be two edge-disjoint matchings of a graph G. If $n \leq |E_2| \leq |E_1| \leq 2n-2$, then there exists $H_1 \subset E_1$ and $G_n \subset E_2$ such that*

$$\begin{aligned} G_n &\cong nK_2, \\ G_{n-1} &= H_1 \cup (E_2 - G_n) \cong (n-1)K_2. \end{aligned}$$

III. Theorem and Corollaries

Theorem *Let G be a graph with size $\binom{n+1}{2}$. If $x'(G) \leq (n+2)/2$, then G has an ascending subgraph decomposition $\{G_i\}, 1 \leq i \leq n$, such that $G_i \cong iK_2$.*

Proof We prove by induction on n. For $n = 2, G$ has desired the ascending subgraph decomposition except $G \cong C_3$. If $G \cong C_3, X'(G) = 3 > \frac{n+2}{2}$. So the result holds. Similarly the result also holds for $n = 3$.

Assume the result holds for $n-2$. We prove it holds for $n \geq 4$.

Let $x'(G) = k$, and let $E_1, E_2, \ldots, E_k$ be the edge color classes in a k-edge-coloring of G. We can assume $|E_1| \geq |E_2| \geq \cdots \geq |E_k|$.

1) If $|E_1| \geq 2n-1$,

$$|E_k| \leq \frac{\binom{n+1}{2} - (2n-1)}{n/2} \leq n-2.$$

By Lemma 2, there exists $H \subset E_1$ such that $G_n = E_k \cup H \cong nK_2$. Define $G_{n-1} \subset E_1 - H$ such that $G_{n-1} \cong (n-1)K_2$.

2) If $|E_1| \leq 2n-2$, we consider three cases.

Case 1 *If $|E_2| = n-1$, we let $G_{n-1} = E_2 \cong (n-1)K_2$. Define $G_n \subset E_1$ such that*

$$G_n \cong nK_2.$$

Case 2 *If $|E_2| \le n-2$, then*

$$|E_1| \ge \binom{n+1}{2} - \frac{n}{2}(n-2) = \frac{3}{2}n.$$

By Lemma 3, there exist $H_1 \subset E_1$ and $H_2 \subset E_2$ such that

$$\begin{aligned} G_n &= H_1 \cup H_2 \cong nK_2, \\ G_{n-1} &= E_1 - H_1 \cong (n-1)K_2. \end{aligned}$$

Case 3 *If $|E_2| \ge n$, then*

$$n \le |E_2| \le |E_1| \le 2n-2.$$

By Lemma 4, there exist $G_n \subset E_2$ and $H_1 \subset E_1$ such that

$$\begin{aligned} G_n &\cong nK_2, \\ G_{n-1} &= H_1 \cup (E_2 - G_n) \cong (n-1)K_2. \end{aligned}$$

In each of Case 1–3, we let $G' = G - G_n - G_{n-1}$. Then $|G'| = \binom{n+1}{2}$, $x'(G') \le \frac{n}{2}$. By hypothesis, G' has an ascending subgraph decomposition $\{G_i\}, 1 \le i \le n-2$, such that $G_i \cong iK_2$.

By the theorem of Vizing[2], for every graph G, either $G \in C'$ or $G \in C^2$. So we have the following corollaries.

Corollary 1 *Let G be a graph with $|G| = \binom{n+1}{2}$. Then G has an ascending subgraph decomposition $\{G_i\}, 1 \le i \le n$, such that $G_i \cong iK_2$, provided $\Delta(G) \le (n+2)/2$ if $x'(G)$ is $\Delta(G)$ or $\Delta(G) \le n/2$ if $x'(G) = \Delta(G)+1$.*

Corollary 2[1] *Let G be a graph with $|G| = \binom{n+1}{2}$, for $n \ge 4$ and $\Delta(G) \le 2$. Then G has an ascending subgraph decomposition $\{G_i\}, 1 \le i \le n$, such that $G_i \cong iK_2$.*

Corollary 3[1] *Let F be a forest with $|F| = \binom{n+1}{2}$, and $n \ge 2\Delta(F) - 2 \ge 2$. Then F has an ascending subgraph decomposition $\{G_i\}, 1 \le i \le n$, such that $G_i \cong iK_2$.*

Corollary 4 *Let G be a bipartite graph with $|G| = \binom{n+1}{2}$, and $n \ge 2\Delta(F) - 2 \ge 2$. Then G has an ascending subgraph decomposition $\{G_i\}, 1 \le i \le n$, such that $G_i \cong iK_2$.*

References

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