Definition 4.2 *Let G be a Lie superalgebra, if G contains no complete proper graded ideals, then G is called a simply complete Lie superalgebra.*

For example, simply Lie superalgebras $A(m,n)(m \neq n), B(m,n), C(n), D(m,n), W(n), \tilde{S}(n)$ are simply complete Lie superalgebras.

From Lemma 4.1, Theorem 2.2 and Definition 4.2, we obtain

Theorem 4.3 *Let G be a complete Lie superalgebra. Then*

i) G is simply complete if and only if G cannot be decomposited into the direct sum of non-trivial graded ideals.

ii) G can be decomposited into the direct sum of simply complete graded ideals.

Acknowledgment *The author wants to thank Professor Chen Zhonghu and Zhang Zhixue for their help.*

References

[1] N. Jacobson, *Lie Algebras*, John Wiley and Sons, INC, 1962.
[2] Meng Daoji, *On Complete Lie Algebras,* Bulletin of Science (in Chinese), Science Press, Beijing, **14**(1985).
[3] M. Scheunert, *The Theory of Lie Superalgebras,* Lecture Notes in Mathematics, 716, Springer-Verlag, 1979.
[4] V.G. Kac, *Lie superalgebras,* Adv. in Math. 26, No.1(1977), 8—96.
[5] J.E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Spring-Verlag, Berlin, 1972.

完备Lie超代数

李振亨
(河北大学数学系，保定071002)

摘 要

本文引入了完备Lie超代数和Lie超代数的全形这两个概念，讨论了完备Lie超代数的一些等价条件和结构定理. 所得结果是Jacobso [1] 和Meng Daoji [2] 的推广.

Journal of Mathematical Research & Exposition
Vol.14, No.2, May 1994

Complete Lie Superalgebras *

Li Zhenheng
(Dept. of Math., Hebei University, Baoding)

Abstract In this paper, we introduce two notions of Complets Lie Superalgebra and the Holomorph of a Lie Superalgebra, and obtain some equivalent conditions for Complete Lie Superalgebras, then we study the struture theorem of Complete Lie Superalgebras.

Keyworks Complete Lie superalgebra, Holomorph, simply complete graded ideal.

1. Preliminaries

Definition 1.1 *Let $G = G_{\bar{0}} \oplus G_{\bar{1}}$ be a superalgebra whose multiplication is denoted by* $[,]$. *This implies in particular that $[G_\alpha, G_\beta] \subset G_{\alpha+\beta}$ for all $\alpha, \beta \in Z_2$. We call G a Lie superalgebra if the multiplication satisfies the following identities*

$$[a,b] = -(-1)^{\alpha\beta}[b,a] \qquad \textit{(graded skew-symmetry)},$$

$$[a,[b,c]] = [[a,b],c] + (-1)^{\alpha\beta}[b,[a,c]] \qquad \textit{(graded Jacobi identity)}$$

for all $a \in G_\alpha, b \in G_\beta, c \in G; \alpha, \beta \in Z_2$.

$G_{\bar{0}}$ is an ordinary Lie algebra, $G_{\bar{1}}$ is a $G_{\bar{0}}$- module.

From now on, if $a \in G_\alpha, \alpha \in Z_2$, then we denote the degree α of a by $\deg a = \alpha$.

Throughout, if $\deg a$ occurs in an expression, then it is assumed that a is homogeneous, and that the expression extend to the other elements by linearity. The base field is the complex field C, and $\dim G < \infty$.

A non empty sabspace K of G is called an ideal if $[a,k] \in K$ for all $a \in G, k \in K$. We shall call G a simple Lie superalgebra if G contains no nontrivial ideals.

A graded subalgebra K (resp. ideal) of G is a subalgebra (resp. ideal) of G and it contains the homogeneous components of all of its elements, i.e.,

$$K = \bigoplus_{\alpha \in Z_2} K \bigcap G_\alpha,$$

so $G/K = G_{\bar{0}}/K \cap G_{\bar{0}} \oplus G_{\bar{1}}/K \cap G_{\bar{1}}$ is the quotient Lie superalgebra of G by K (when K is a graded ideal of G).

*Received Oct. 8, 1991. Project Supported by the Hebei Provice Science Foundation, No.193005.

The centralizer of a subset X of G is $C_G(X)=\{x\in G|[x,X]=0\}.C_G(G)$ is called the centre of G and denoted by $C(G)$.

A linear transformation $\varphi: G\to G'(G=G_{\bar 0}\oplus G_{\bar 1}, G'=G'_{\bar 0}\oplus G'_{\bar 1}$ are Lie superalgebras) is called a homomorphism if $\varphi(G_{\bar 0})\subseteq G'_{\bar 0}, \varphi(G_{\bar 1})\subseteq G'_{\bar 1}$ and $\varphi[x,y]=[\varphi(x),\varphi(y)]$, for all $x,y\in G;\varphi$ is called a monomorphism if Ker $\varphi=0$, an epimorphism if Im $\varphi=G'$, an isomorphism if it is both mono- and epi-. If G is isomorphic to G', we denote by $G\approx G'$.

A linear mapping $g: G\to G'$ is said to be homogeneous of degree $s, s\in Z_2$, if

$$g(G_\alpha)\subseteq G'_{\alpha+s}\quad \text{for all } \alpha\in Z_2.$$

We denote by $\mathrm{End}_s(G)$ the set of all linear mapping $g: G\to G$ of homogeneous of degree $s, s\in Z_2$, i.e., $\mathrm{End}_s(G)=\{g: G\to G\mid g(G_\alpha)\subseteq G_{\alpha+s}, \alpha\in Z_2\}$.

A derivation of degree $s(s\in Z_2)$ of G is an element $D\in\mathrm{End}_s(G)$ with the property

$$D[a,b]=[D(a),b]+(-1)^{s\deg a}[a,D(b)] \text{ for all } a,b\in G.$$

The set of all derivations of degree s is denoted by $\mathrm{der}_s(G)\subseteq \mathrm{End}_S(G)$.

Put $\mathrm{der}(G)=\mathrm{der}_{\bar 0}(G)+\mathrm{der}_{\bar 1}(G)$. It is closed under $[\delta_1,\delta_2]=\delta_1\delta_2-(-1)^{\deg\delta_1\deg\delta_2}\delta_2\delta_1$, $\mathrm{der}(G)$ is called the Lie superalgebra of derivations of G. It is easy to see that

$$\mathrm{der}(G)=\mathrm{der}_{\bar 0}(G)\oplus\mathrm{der}_{\bar 1}(G)$$

is a direct sum.

If $a\in G$ we define the linear map $ada: G\to G$ by $ada(b)=[a,b]$ for all $b\in G$, it follows from the graded Jacobi identity that ada is a derivation of G, for all $a\in G$. It is clear that ad is a homomorphism of G into Lie superalgebra $\mathrm{der}(G)$. The derivations of G which are of the forms $ada, a\in G$, are called inner. Let $ad(G)=\{ada|a\in G\}$, then $ad(G)$ is an ideal of $\mathrm{der}(G)$. One easily checks that $ada\in\mathrm{der}_s(G)$ if and only if $a\in G_s, s\in Z_2; a\in G$, hence $ad(G)=ad_{\bar 0}(G)\oplus ad_{\bar 1}(G)$.

2. On complete Lie superalgebras

Definition 2.1 *We shall call a Lie superalgebra complete if its derivations are all inner and its centre is 0.*

Example 1 If G is one of the classical Lie superalgebras $A(m,n),\ m\neq n, B(m,n), C(n), D(m,n), F(4), G(3)$ then G is a complete Lie superalgebra.

Example 2 If G is one of then $W(n)$ and $\tilde S(n)$, then G is a complete Lie superalgebra.

The following theorem is the main result in this section, which is the generalization of Jacobson [1].

Theorem 2.2 *If K is complete and a graded ideal in G, then $G=K\oplus B$, where B is a graded ideal of G.*

Proof We note first that if K is a graded ideal in G, then the centralizer B of K is a graded ideal. $B=\{b\in G\mid [b,K]=0\}$ is evidently a subspace. For any $b\in B\subseteq G=G_{\bar 0}\oplus G_{\bar 1}$, let $b=b_0+b_1, b_0\in G_{\bar 0}, b_1\in G_{\bar 1}$, then $[b_0+b_1,k]=0$ for all $k\in K=K\cap G_{\bar 0}\oplus K\cap G_{\bar 1}$, hence $[b_0, k] = [b_1, k] = 0$ for all $k \in K$. Thus $b_0 \in B, b_1 \in B$, we have $B = B_0 + B_1$ where $B_{\bar{i}} = B \cap G_{\bar{i}}$ $(i = 0, 1)$, hence B is a graded subspace. Now, if $b \in B$ and $a \in G$, then $[a, [b, k]] = [[a, b], k] + (-1)^{\deg a \deg b}[b, [a, k]]$, i.e., $0 = [[a, b], k] + 0$, we obtain $[[a, b], k] = 0$ and $[a, b] \in B$. Hence B is a graded ideal in G.

Let K be complete, if $c \in K \cap B$, then c is in the center of K and so $c = 0$. Hence $K \cap B = 0$. Next let $a \in G$, since K is an ideal in G, ada maps K into itself and hence it induces a derivation in K. This is inner and so we have a $k \in K$ such that $ada\,|_K = adk$. Let $a = a_0 + a_1, k = k_0 + k_1$, where $a_i \in G_{\bar{i}}, b_i \in K \cap G_{\bar{i}}$, we obtain

$$ada_0 + ada_1 = adk_0 + adk_1,$$

then $ada_i = adk_i (i = 0, 1)$, hence $[a_0, x] = [k_0, x], [a_1, x] = [k_1, x]$ for all $x \in K$. Take $b_i = a_i - k_i$, so $b_i \in B_{\bar{i}} (i = 0, 1)$, let $b = b_0 + b_1$ so we have $a = b + k, b \in B, k \in K$. Thus $G = K + B = K \oplus B$ as required.

Theorem 2.3 *Let K be a Lie superalgebra, if any Lie superalgebra G for which K is its ideal has a decomposition of ideals $G = K \oplus C_G(K)$, then K is a complete Lie superalgebra.*

The proof follows immediately from Theorem 3.4 in the following section.

3. The holomorph of Lie superalgebras

Let $H(G) = G \oplus \text{der}(G), I(G) = G \oplus ad(G)$ where $\oplus$ is the direct sum of vector spaces, then

$$\begin{aligned} H(G) &= H_{\bar{0}}(G) \oplus H_{\bar{1}} \text{ where } H_\alpha(G) = G_\alpha \oplus \text{der}_\alpha(G), \alpha \in Z_2, \\ I(G) &= I_{\bar{0}}(G) \oplus I_{\bar{1}} \text{ where } I_\alpha(G) = G_\alpha \oplus ad_\alpha(G), \alpha \in Z_2 \end{aligned}$$

are graded vector spaces.

Lemma 3.1 *Under the above notations, if we define a bracket as follows*

$$[x + D, y + E] = [x, y] + Dy - (-1)^{degE\, degx} Ex + [D, E]$$

for all $x, y \in G = G_{\bar{0}} \oplus G_{\bar{1}}; D, E \in der(G)$ (resp. $ad(G)$), then $H(G)$ (resp. $I(G)$) is a Lie superalgebra.

Proof It is clear that $H(G)$ is a Z_2-graded space and $[H_\alpha(G), H_\beta(G)] \subseteq H_{\alpha+\beta}(G)$, where $\alpha, \beta \in Z_2$.

Now $[y + E, x + D] = [y, x] + Ex - (-1)^{\deg D \deg y} Dy + [E, D]$, we can check that $[y + E, x + D] = (-1)(-1)^{\deg(x+D)\deg(y+E)}[x + D, y + E]$ case by case. For example, when $\deg(x + D) = \deg(y + E) = 1$, we have

$$\begin{aligned} [x + D, y + E] &= [x, y] + Dy + Ex + [D, E], \\ [y + E, x + D] &= [y, x] + Ex + Dy + [E, D], \end{aligned}$$

since $[x, y] = [y, x]$ in $G, [D, E] = [E, D]$ in $\text{der}(G)$ (here $\deg x = \deg y = 1$, so $[x, y] = [y, x]$ in G, $[D, E] = [E, D]$ is similar in $\text{der}(G)$), so

$$[x + D, y + E] = (-1)(-1)^{\deg(x+D)\deg(y+E)}[y + E, x + D] \text{ in } H(G).$$

Next, we will check the graded Jacobi identity in $H(G)$ case by case.

When $\deg(a+D) = \deg(b+E) = \deg(c+F) = 1$

$$3.1)\quad [a+D,[b+E,c+F]] = [a+D,[b,c]+Ec-(-1)^{\deg F\deg b}Fb+[E,F]]$$
$$= [a,[b,c]]+[a,Ec]+[a,Fb]+D[b,c]+DEc+DFb-[E,F]a+[D,[E,F]],$$

$$3.2)\quad [[a+D,b+E],c+F] = [[a,b]+Db-(-1)^{\deg E\deg a}Ea+[D,E],c+F]$$
$$= [[a,b],c]+[Db,c]+[Ea,c]+[D,E]c-F[a,b]-FDb-FEa+[[D,E],F].$$

The difference 3.1)—3.2) is

$$\begin{aligned} &(-1)[b,[a,c]]+[a,Ec]-[Ea,c]+[a,Fb]+F[a,b]+D[b,c]-[Db,c]\\ &\quad +DEc-[D,E]c+DFb+FDb-[E,F]a+FEa+(-1)[E,[D,F]]\\ =\ & -[b,[a,c]]-E[a,c]+[Fa,b]-EDc+[D,F]b-EFa-[E,[D,F]]\\ =\ & (-1)^{\deg(a+D)\deg(b+E)}[b+E,[a+D,c+F]]. \end{aligned}$$

Hence, graded Jacobi identity holds (the other cases are similar). Thus, $H(G)$ is a Lie superalgebra. Similarly, $I(G)$ is a Lie superalgebra.

Definition 3.2 *The Lie superalgebra $H(G)$ and $I(G)$ are called holomorph and inner holomorph of G respectively.*

It is clear that G and $I(G)$ are graded ideals of $H(G)$; $H(G)/G$ is isomorphic to $\mathrm{der}(G) = H_{\bar{0}}/G_{\bar{0}} \oplus H_{\bar{1}}/G_{\bar{1}}$.

Evidently, G is complete if and only if $H(G) = I(G)$.

Lemma 3.3 *Under the above notations, we have*

i) $C_{H(G)}(G) = \{x - adx \mid \forall x \in G\}$ *is a graded ideal of* $H(G)$,

ii) $G \cap C_{H(G)}(G) = C(G)$ *(is a graded ideal of* G*),*

iii) The map θ of $H(G)$ into $H(G)$ defined by the following $\theta : x + D \mapsto adx - x + D$ is an isomorphism of $H(G)$ such that $\theta^2 = id, \theta(G) = C_{H(G)}(G)$.

Proof One can check directly that both (i) and (ii) are true (c.f. Lemma 5 in [2]).

For (iii), one easily sees that θ is linear, $\theta^2 = id$ and $\theta(G) = C_{H(G)}(G)$, hence, we must prove that θ preserve the bracket, that is $[\theta(x+D),\theta(y+E)] = \theta([x+D,y+E])$.

When $\deg(x+D) = \deg(y+E) = 1$

$$\begin{aligned} &[\theta(x+D),\theta(y+E)] = [adx - x + D, ady - y + E]\\ =\ & [x,y] - adx(y) - Dy - (-1)^{\deg(ady+E)\deg x}(ady+E)(-x) + [adx+D, ady+E]\\ =\ & -Dy - (-1)^{\deg(ady+E)\deg x} ady(-x) + E(-x) + [adx, ady]\\ &\quad + [adx, E] + [D, ady] + [D, E]\\ =\ & ad([x,y]+Dy+Ex) - ([x,y]+Dy+Ex) + [D,E]\\ =\ & \theta([x+D,y+E]). \end{aligned}$$

The other cases

a) $\deg x = \deg(adx) = 0$, $\deg D = 0$, $\deg y = \deg(ady) = 0$, $\deg E = 0$,

b) $\deg x = \deg(adx) = 0$, $\deg D = 0$, $\deg y = \deg(ady) = 1$, $\deg E = 1$

are similar, we omitted the proof here.

Now, we obtain the main theorem of this section.

Theorem 3.4 *Let G be a Lie superalgebra, and the holomorph of G has a decomposition of graded ideals $H(G) = G \oplus C_{H(G)}(G)$, then G is a complete Lie superalgebra.*

Proof By ii) of Lemma 3.3 we have $C(G) = G \cap C_{H(G)}(G) = 0$. Next

$$\mathrm{der}(G) \approx H(G)/G \approx C_{H(G)}(G) = \theta(G) \approx G,$$

but $ad(G) \approx G$ (since $C(G) = 0$), so $\mathrm{der}(G) \approx ad(G)$. Since $ad(G)$ is an ideal of $\mathrm{der}(G)$, hence $ad(G) = \mathrm{der}(G)$.

4. The decomposition of complete Lie superalgebras

Lemma 4.1 *Let $G = K_1 \oplus K_2, K_i (i = 1,2)$ are graded ideals in G. Then we have*

(i) if $C(G) = 0$ then $ad(G) = ad(K_1) \oplus ad(K_2)$, $der(G) = der(K_1) \oplus der(K_2)$;

(ii) G is complete if and only if K_1 and K_2 are complete, moreover, if G and K_1 are complete then K_2 is complete.

Proof For any $D \in \mathrm{der}(K_1)$, let $Dx_2 = 0$ for all $x_2 \in K_2$, then $D \in \mathrm{der}(G)$, so we regard $\mathrm{der}(K_1) \subseteq \mathrm{der}(G)$, and similarly $\mathrm{der}(K_2) \subseteq \mathrm{der}(G)$. Evidently, $D \in \mathrm{der}(K_1)$ if and only if $Dx_2 = 0, \forall x_2 \in K_2; D \in \mathrm{der}(K_2)$ if and only if $Dx_1 = 0, \forall x_1 \in K_1$, so

$$\mathrm{der}\ (K_1) \bigcap \mathrm{der}\ (K_2) = 0.$$

Now, for any $D_1 \in \mathrm{der}(K_1), D \in \mathrm{der}(G), x_2 \in K_2$, one have

$$[D, D_1]x_2 = (DD_1 - (-1)^{\deg D_1 \deg D} D_1 D)(x_2) = (-1)(-1)^{\deg D_1 \deg D} D_1 D x_2,$$

but, for any $x_i \in K_i (i = 1,2), D \in \mathrm{der}(G)$, one have

$$\begin{aligned} [Dx_1, x_2] &= D[x_1, x_2] - (-1)^{\deg D \deg x_1}[x_1, Dx_2] \\ &= -(-1)^{\deg D \deg x_1}[x_1, Dx_2] \in K_1 \bigcap K_2, \end{aligned}$$

so $[Dx_1, x_2] = [x_1, Dx_2] = 0$, thus $Dx_i \in K_i (i = 1,2)$, hence $[D, D_1] \in \mathrm{der}(K_1)$, that is, $\mathrm{der}(K_1)$ is an ideal in $\mathrm{der}(G)$. Similarly, $\mathrm{der}(K_2)$ is an ideal in $\mathrm{der}(G)$.

Thus we know that there exist $D_i \in \mathrm{der}(K_i) (i = 1,2)$ such that $[D = D_1 + D_2$, so $\mathrm{der}(G) = \mathrm{der}(K_1) \oplus \mathrm{der}(K_2)$ (hence $ad(G) = ad(K_1) \oplus ad(K_2)$).

Use (i) we know that (ii) is clear.

Definition 4.2 *Let G be a Lie superalgebra, if G contains no complete proper graded ideals, then G is called a simply complete Lie superalgebra.*

For example, simply Lie superalgebras $A(m,n)(m \neq n), B(m,n), C(n), D(m,n), W(n)$, $\tilde{S}(n)$ are simply complete Lie superalgebras.

From Lemma 4.1, Theorem 2.2 and Definition 4.2, we obtain

Theorem 4.3 *Let G be a complete Lie superalgebra. Then*

i) G is simply complete if and only if G cannot be decomposited into the direct sum of non-trivial graded ideals.

ii) G can be decomposited into the direct sum of simply complete graded ideals.

Acknowledgment *The author wants to thank Professor Chen Zhonghu and Zhang Zhixue for their help.*

References

[1] N. Jacobson, *Lie Algebras*, John Wiley and Sons, INC, 1962.
[2] Meng Daoji, *On Complete Lie Algebras,* Bulletin of Science (in Chinese), Science Press, Beijing, **14**(1985).
[3] M. Scheunert, *The Theory of Lie Superalgebras,* Lecture Notes in Mathematics, 716, Springer-Verlag, 1979.
[4] V.G. Kac, *Lie superalgebras,* Adv. in Math. 26, No.1(1977), 8—96.
[5] J.E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Spring-Verlag, Berlin, 1972.

完备Lie超代数

李振亨
(河北大学数学系，保定071002)

摘　要

本文引入了完备Lie超代数和Lie超代数的全形这两个概念，讨论了完备Lie超代数的一些等价条件和结构定理. 所得结果是Jacobso [1] 和Meng Daoji [2] 的推广.