第14卷第2期　　数 学 研 究 与 评 论　　Vol. 14 No. 2
1994年5月　　JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION　　May 1994

Opial—Olech 型不等式的改进及其应用*

胡　　克
(江西师范大学数学系，南昌 330027)

摘　要　本文给出：设 $f(x)$ 在 $[0,h]$ 上绝对连续. $f(0)=f(h)=0$, $p>0$, $q>1$ 和 $s=p/(p+q-1)$，则有

$$\int_0^h |f|^p |f'|^q dx \leqslant \frac{1}{(p+q)^s}\left(\frac{h}{2}\right)^p\left(\int_0^h |f'|^{p+q}dx\right)$$
$$\cdot\left\{1-\frac{1}{4}\left[\left(\int_0^h \cos\left(\frac{2\pi x}{h}\right)dx\,|f'|^{p+q}dx\right)/\left(\int_0^h |f'|^{p+q}dx\right)\right]^2\right\}^{\theta(p)} \tag{A}$$

其中 $\theta(p)=\frac{1}{2}s$ 当 $p+q>0$, $\theta(p)=\frac{p}{2}$，当 $1<p+q<2$.

若代(A)右边为零. 即为 Opial—Olech 不等式. 实际上本文所得结果还要广泛.

关键词：　绝对连续

§1　引　言

定理 A　若 $f'(x)$ 为 $[0,h]$ 上的连续函数，$f(0)=f(h)=0$，则

$$\int_0^h |f(x)f'(x)|dx \leqslant \frac{h}{4}\int_0^h |f'(x)|^2 dx \tag{B}$$

后来 C. Olech[2] 证明：以 f 为绝对连续函数代替 $f'(x)$ 为连续. (B)亦成立. Opial，Olech 的工作发表以后，引起了不少数学家的兴趣，导致一系列有关文章发表. 在中国，著名数学家华罗庚先生首先在中国科学[3]发表文章将(B)式加以推广. 然而无论在国内和国外从有关文献看，可以断言：已发表的有关 Opial—Olech 不等式一系列文章，只是 Opial—Olech 所得不等式(1)的各种形式的推广，无实质改进意义. 本文目的在

1. 对 Opial—Olech 所得不等式(1)给予实质上的改进. 并给出应用.
2. 对华罗庚推广之 Opial—Olech 型不等式给予实质上的改进.

§2　Opital—Olech—Beesack 不等式的改进

定理 1　设(i) $f(x)$ 在 $[0,h]$ 上绝对连续，$f(0)=f(h)=0$. 若(ii) $B(x)>0$, $\int_0^{h/2}\frac{1}{B(x)}dx=\int_{h/2}^h\frac{1}{B(x)}dx=k_1$,　(iii) $1-e(x)+e(y)\geqslant 0$, $\int_0^{h/2}\frac{e(x)}{B(x)}dx=\int_{h/2}^h\frac{e(x)}{B(x)}dx=k_2$，则有

$$\int_0^h |f(x)f'(x)|dx \leqslant \frac{1}{2}\left\{\left(k_1\int_0^h B(x)|f'(x)|^2dx\right)^2-w^2(c,h)\right\}^{1/2}, \tag{1}$$

* 1992年11月28日收到. 国家和省自然科学基金资助课题.

其中 $k_1 w(0,h) = k_1\int_0^h e(x)B(x)|f'(x)|^2dx - k_2\int_0^h B(x)|f'(x)|^2dx$. 特别有

$$\int_0^h|f(x)f'(x)|dx \leqslant \frac{h}{4}\{(\int_0^h|f'(x)|^2dx)^2 - (\frac{1}{2}\int_0^h|f'(x)|^2\cos(\frac{2\pi x}{h})dx)^2\}^{1/2}. \tag{2}$$

若取 $B(x)=1, e(x)=\frac{1}{2}\cos(\frac{2\pi x}{n})$,(1)即为(2). 若取 $B(x)\equiv e(x)\equiv 1$,则 $w(0,h)=0$,即为 Opial－Olech 不等式.

证明 我们只要证明(2)就可以了. 证明前先述证二个引理:

引理 1[4] 设 $P\geqslant Q>1, \frac{1}{P}+\frac{1}{Q}=1$ 及 $F(x), G(x)\geqslant 0$,则有

$$\int_a^b F(x)G(x)dx \leqslant (\int_a^b G^Q(x)dx)^{\frac{1}{Q}-\frac{1}{P}}\{(\int_a^b F^P(x)dx\int_a^b G^Q(x)dx)^2 - w_1^2(a,b)\}^{\frac{1}{2P}}, \tag{3}$$

其中

$$w_1(a,b) = \int_a^b e(x)F^P(x)dx\int_a^b G^Q(x)dx - \int_a^b F^P(x)dx\int_a^b e(x)G^Q(x)dx.$$

引理 2 设 $g(x)$ 为$[0,a]$上的绝对连续函数, $g(0)=0, B(x)>0$. $\int_0^a\frac{dx}{B(x)}$ 存在和 $1-e(x)+e(y)\geqslant 0$,则

$$\int_0^a|g(x)g'(x)|dx \leqslant \frac{1}{2}\{(\int_0^a\frac{dx}{B(x)}\int_0^a B(x)|g'(x)|^2dx)^2 - w_2^2(0,a)\}^{1/2}, \tag{4}$$

其中 $w_2(0,a) = \int_0^a\frac{dx}{B(x)}\int_0^a e(x)B(x)|g'(x)|^2dx - \int_0^a\frac{e(x)}{B(x)}dx\int_0^a B(x)|g'(x)|^2dx$.

此为 Olech 不等式的改进.

证明 设 $h(x) = \int_0^x|g'(x)|dx,\ x\in[0,a]$, 则 $|g(x)|\leqslant h(x)$. 又

$$\int_0^a|g(x)g'(x)|dx \leqslant \int_0^a h(x)h'(x)dx = \frac{1}{2}h^2(a) = \frac{1}{2}(\int_0^a g'(x)dx)^2. \tag{5}$$

在引理 1 中,取 $P=Q=2, F(x)=\sqrt{B(x)}h'(x), G(x)=1/\sqrt{B(x)}$. 则有

$$\begin{aligned} h^2(a) = (\int_0^a|g'(x)|dx)^2 &= \int_0^a\sqrt{B(x)}|g'(x)|\cdot\frac{1}{\sqrt{B(x)}}dx \\ &\leqslant \{(\int_0^a\frac{dx}{B(x)}\int_0^a B(x)|g'(x)|^2dx)^2 - (\int_0^a\frac{e(x)}{B(x)}dx\int_0^a B(x)|g'(x)|^2dx \\ &\quad - \int_0^a\frac{dx}{B(x)}\int_0^a e(x)B(x)|g'(x)|^2dx)^2\}^{1/2}. \end{aligned} \tag{6}$$

将(6)代入(5)即得(4)式,证毕.

定理 1 的证明 由假设及引理 1,得

$$I_1 = \int_0^{h/2}|f(x)f'(x)|dx \leqslant \frac{k_1}{2}\{(\int_0^{h/2}B(x)|f'(x)|^2dx)^2 - w^2(0,h/2)\}^{1/2}, \tag{7}$$

$$I_2 = \int_0^{h/2}|f(h-x)f'(h-x)|dx \leqslant \frac{k_1}{2}\{(\int_0^{h/2}B(h-x)|f'(h-x)|^2dx)^2 - w^2(0,h/2)\}^{1/2}. \tag{8}$$

I_2 即可写成

$$I_2 = \int_{h/2}^h|f(x)f'(x)|dx \leqslant \frac{k_1}{2}\{(\int_{h/2}^h B(x)|f'(x)|^2dx)^2 - w^2(h/2,h)\}^{1/2}. \tag{8'}$$

注意到下面简单的事实:若 $A_+, A_-\geqslant 0$ 和 $B_+, B_-\geqslant 0$ 则

$$\sqrt{A_+A_-} + \sqrt{B_+B_-} \leqslant \sqrt{(A_+ + B_+)(A_- + B_-)}. \tag{9}$$

令

$$A_+ = \int_0^{h/2} B(x)|f'(x)|^2 dx + w(0,h/2), \qquad A_- = \int_0^{h/2} B(x)|f'(x)|^2 dx - w(0,h/2);$$

$$B_+ = \int_{h/2}^{h} B(x)|f'(x)|^2 dx + w(h/2,h), \qquad B_- = \int_{h/2}^{h} B(x)|f'(x)|^2 dx - w(h/2,h).$$

则由(7),(8)和(9),有

$$\int_0^h |f(x)f'(x)|dx = I_1 + I_2 \leqslant \frac{k_1}{2}\{\sqrt{A_+A_-} + \sqrt{B_+B_-}\} \leqslant \frac{1}{2}\sqrt{(A_+ + B_+)(A_- + B_-)}$$

$$= \frac{k_1}{2}\{(\int_0^h B(x)|f'(x)|^2 dx)^2 - w^2(0,h)\}^{1/2}. \tag{10}$$

§3 华罗庚—Opial 型不等式的改进

定理 2 设 $p\geqslant 0, q\geqslant 1, s=p/(p+q-1)$,又设 $f(x)$ 为 $[a,b]$ 上的绝对连续函数 $f(a)=0$,则

$$\int_a^b |f(x)|^p|f'(x)|^q dx \leqslant \begin{cases} (\frac{1}{p+q})^s (b-a)^p (\int_a^b |f'(x)|^{p+q}dx)^{1-s}\{(\int_a^b |f'(x)|^{p+q}dx)^2 \\ \quad - w^2(a,b)\}^{s/2}, p+q \geqslant 2; & (11) \\ (\frac{1}{p+q})^s (b-a)^p (\int_a^b |f'(x)|^{p+q}dx)^{1-p}\{(\int_a^b |f'(x)|^{p+q}dx)^2 \\ \quad - w^2(a,b)\}^{p/2}, 1 < p+q < 2. & (12) \end{cases}$$

其中 $(b-a)w(a,b) = \int_a^b e(x)dx \int_a^b |f'|^{p+q}dx - (b-a)\int_a^b |f'|^{p+q}e(x)dx$, $1 - e(x) + e(y) \geqslant 0$, x, $y \in [a,b]$.

若 $q=1$,见[6].若取 $e(x)\equiv 1$,当 $q=1$,p 为正整数首先为华罗庚所得.当 $q>1$,$p>0$ 为 G. S. Yang[5]所得.注意 $p\geqslant 0, q\geqslant 1, (\frac{1}{p+q})^s \leqslant \frac{q}{p+q}$.

证明 同引理 2 的证明,记 $h(x) = \int_a^x |f'(x)|dx$,所以可得

$$\int_a^b |f|^{p+q-1}|f'|dx \leqslant \int_a^b h^{p+q-1}h'dx = \frac{1}{p+q}h^{p+q}(b). \tag{13}$$

同引理 1,可得:

$$h(b) = \int_a^b |f'(x)|dx \leqslant \begin{cases} (b-a)^{\frac{p+q-1}{p+q}}\{(\int_a^b |f'|^{p+q}dx)^2 - w^2(a,b)\}^{\frac{1}{2(p+q)}}, p+q \geqslant 2, & (14) \\ (b-a)^{\frac{p+q-1}{p+q}}\{(\int_a^b |f'|^{p+q}dx)^2 - w^2(a,b)\}^{\frac{p+q-1}{2(p+q)}} \\ \quad \cdot (\int_a^b |f'|^{p+q}dx)^{\frac{2-p-q}{p+q}}, 1 \leqslant p+q < 2. & (15) \end{cases}$$

又由 Hölder 不等式,得

$$\int_a^b |f|^p|f'|^q dx = \int_a^b |f|^p|f'|^s|f'|^{q-s}dx \leqslant (\int_a^b |f|^{p+q-1}|f'|dx)^s(\int_a^b |f'|^{p+q}dx)^{1-s}$$

$$\leqslant \frac{1}{(p+q)^s}\left(\int_a^b |f'|dx\right)^{s(p+s)}\left(\int_a^b |f'|^{p+q}dx\right)^{1-s}. \tag{16}$$

此处用到(13)式.

将(14)和(15)分别代入(16)式,便得定理的证明.

定理 3 同定理 3 的假设,再设 $f(b)=0,\int_{\frac{a+b}{2}}^b e(x)dx=\int_a^{\frac{a+b}{2}} e(x)dx=k_2$.

则有

$$\int_a^b |f|^p|f'|^q dx \leqslant \begin{cases} \left(\frac{1}{p+q}\right)^s\left(\frac{b-a}{2}\right)^p\left(\int_a^b |f'|^{p+q}dx\right)^{1-s}\left\{\left(\int_a^b |f'|^{p+q}dx\right)^2 - w_1^2(a,b)\right\}^{s/2}, p+q\geqslant 2; & (17) \\ \left(\frac{1}{p+q}\right)^s\left(\frac{b-a}{2}\right)^p\left(\int_a^b |f'|^{p+s}dx\right)^{1-p}\left\{\left(\int_a^b |f'|^{p+q}dx\right)^2 - w_1^2(a,b)\right\}^{p/2}, 1\leqslant p+q<2. & (18) \end{cases}$$

其中 $\left(\frac{b-a}{2}\right)w_1(a,b)=\left(\frac{b-a}{2}\right)\int_a^b |f'|^{p+q}e(x)dx-k_2\int_a^b |f'|^{p+q}dx$.

证明 由定理 2,得:

$$I_1=\int_a^{\frac{a+b}{2}} |f|^p|f'|^q dx \leqslant \begin{cases} \left(\frac{1}{p+q}\right)^s\left(\frac{b-a}{2}\right)^p\left(\int_a^{\frac{a+b}{2}} |f'|^{p+q}dx\right)^{1-s}\left\{\left(\int_a^{\frac{a+b}{2}} |f'|^{p+q}dx\right)^2 - w^2\left(a,\frac{a+b}{2}\right)\right\}^{s/2}, p+q\geqslant 2; & (19) \\ \left(\frac{1}{p+q}\right)^s\left(\frac{b-a}{2}\right)^p\left(\int_a^{\frac{a+b}{2}} |f'|^{p+s}dx\right)^{1-p}\left\{\left(\int_a^{\frac{a+b}{2}} |f'|^{p+q}dx\right)^2 - w^2\left(a,\frac{a+b}{2}\right)\right\}^{p/2}, 1<p+q<2. & (20) \end{cases}$$

同证明定理 1 的方法可知.

$$I_2=\int_{\frac{a+b}{2}}^b |f|^p|f'|^q dx \leqslant \begin{cases} \left(\frac{1}{p+q}\right)^s\left(\frac{b-a}{2}\right)^p\left(\int_{\frac{a+b}{2}}^b |f'|^{p+q}dx\right)^{1-s}\left\{\left(\int_{\frac{a+b}{2}}^b |f'|^{p+q}dx\right)^2 - w_1^2\left(\frac{b+a}{2},b\right)\right\}^{s/2}, p+q\geqslant 2; & (21) \\ \left(\frac{1}{p+q}\right)^s\left(\frac{b-a}{2}\right)^p\left(\int_{\frac{a+b}{2}}^b |f'|^{p+s}dx\right)^{1-p}\left\{\left(\int_{\frac{a+b}{2}}^b |f'|^{p+q}dx\right)^2 - w_1^2\left(\frac{b+a}{2},b\right)\right\}^{p/2}, 1\leqslant p+q<2. & (22) \end{cases}$$

为方便先证 $1\leqslant p+q<2$ 的情形定理成立. 注意:对任意 $A,B>0$ 的实数,$a\in(0,1)$ 及 $t>0$ 有 $A^aB^{1-a}\leqslant aAt^{-\frac{1}{a}}+(1-a)\beta t^{\frac{1}{1-a}}$. 那么由(20)和(22)就得到

$$\int_a^b |f|^p|f'|^q dx = I_1+I_2 \leqslant \left(\frac{1}{p+q}\right)^s\left(\frac{b-a}{2}\right)^p\left\{(1-p)t^{\frac{1}{1-p}}\left(\int_a^{\frac{a+b}{2}} |f'|^{p+q}dx\right.\right.$$

$$\left.\left.+\,pt^{-\frac{1}{p}}\left[\left(\int_a^{\frac{a+b}{2}} |f'|^{p+q}dx\right)^2 - w_1^2\left(a,\frac{a+b}{2}\right)\right]\right\}^{1/2}\right\}$$

$$+\left(\frac{1}{p+q}\right)^s\left(\frac{b-a}{2}\right)^p\left\{(1-p)t^{\frac{1}{1-p}}\int_{\frac{a+b}{2}}^b |f'|^{p+q}dx\right.$$

$$+ pt^{-\frac{1}{p}}[(\int_{\frac{a+b}{2}}^{b}|f'|^{p+q}dx)^2 - w^2(\frac{a+b}{2},b)]\}^{1/2}\}$$

$$\leqslant (\frac{1}{p+q})^q(\frac{b-a}{2})^p\{(1-p)t^{\frac{1}{1-p}}\int_a^b|f'|^{p+q}dx$$

$$+ pt^{-\frac{1}{p}}[(\int_a^{\frac{a+b}{2}}|f'|^{p+q}dx)^2 - w^2(a,\frac{a+b}{2})]^{1/2}$$

$$+ pt^{-\frac{1}{p}}[(\int_{\frac{a+b}{2}}^{b}|f'|^{p+q}dx)^2 - w^2(\frac{a+b}{2},b)]^{1/2} \tag{23}$$

$$\leqslant (\frac{1}{p+q})^q(\frac{b-a}{2})^p\{(1-p)t^{\frac{1}{1-p}}(\int_a^b|f'|^{p+q}dx$$

$$+ pt^{-\frac{1}{p}}[(\int_a^b|f'|^{p+q}dx)^2 - w^2(a,b)]^{1/2} \tag{24}$$

以上从(23)过渡到(24)式. 在(23)中{}内最后两项用到了(9)式. 在(25)中取 $t^{\frac{1}{(1-p)}} = [(\int_a^b|f'|)^{p+q}dx)^2 - w_1^2(a,b)]^{1/2}/\int_a^b|f'|^{p+q}dx$ 立得: $1\leqslant p+q<2$ 时定理的证明.

同样方法步骤可得 $p+q\geqslant 2$ 时定理的证明.

若取 $a=0, b=h, e(x)=\frac{1}{2}\cos(\frac{2\pi x}{n})$, 就有 $w(0,h) = \frac{1}{2}\int_0^h\cos(\frac{2\pi x}{n})|f'|^{p+q}dx$.

§3 应用

定理 4 设 $a_n(n=1,2,\cdots)$ 为任意实数, 则

$$|\sum a_n|^2 + |\sum(-1)^n a_n|^2 \leqslant (\frac{\pi}{2})^2\{\sum n^2a_n^2)^2 - \frac{1}{4}(\sum n(n+1)a_na_{n+1})^2\}^{1/2}. \tag{25}$$

证明 设 $f(x) = \sum a_n\cos nx$. 因 $\int_0^\pi f(x)dx = 0$, 所以必有 $c\in(a,b)$ 使 $f(c)=0$. 因此

$$|f^2(\pi)| = 2|\int_c^\pi f(x)f'(x)dx| \leqslant 2\int_c^\pi|f||f'|dx,$$

$$|f^2(0)| = 2|\int_0^c f(x)f'(x)dx| \leqslant 2\int_0^c|f||f'|dx.$$

即得 $|f^2(0)| + |f^2(\pi)| \leqslant 2\int_0^\pi|f||f'|dx$.

由改进后的 Olech 不等式(即引理 2). 我们有

$$|f^2(0)| + |f^2(\pi)| \leqslant \{(\pi\int_0^\pi|f'(x)|^2dx)^2 - (\pi\int_0^\pi|f'(x)|^2e(x)dx$$

$$- \int_0^\pi e(x)dx\int_0^\pi|f'(x)|^2dx)^2\}^{1/2}. \tag{26}$$

在(26)取 $e(x)=\frac{1}{2}\cos x$. 通过计算得:

$$\int_0^\pi|f'(x)|^2dx = \frac{\pi}{2}\sum n^2a_n^2, \tag{27}$$

$$\int_0^\pi|f'(x)|^2\cos x dx = \frac{\pi}{2}\sum n(n+1)a_na_{n+1}, \tag{28}$$

$$\int_0^{\pi} \cos x dx = 0. \tag{29}$$

将(27),(28)和(29)代入(28)中即得定理.

定理 5 设 $a_n \geqslant 0$,我们有

$$\left(\sum_{n\geqslant 1} a_n\right)^2 \leqslant \frac{\pi^2}{2}\left\{\left(\sum_{n\geqslant 1}\left(n-\frac{1}{2}\right)^2 a_n^2\right)^2 - \left(\sum_{n\geqslant 1}\left(n^2-\frac{1}{4}\right)a_n a_{n+1}\right)^2\right\}^{1/2}. \tag{30}$$

证明 在引理 2 中取 $f(x)=\sum_{n\geqslant 1} a_n \cos(2n-1)x$, $f(\frac{\pi}{2})=0$. 可取 $b=\pi/2, a=0, B(x)=1$ 及 $C(x)=\frac{1}{2}\cos 2x$,则

$$\left(\sum a_n\right)^2 = f^2(0) \leqslant 2\int_0^{\frac{\pi}{2}} |f||f'|dx \leqslant 2\left\{\left(\frac{\pi}{2}\int_0^{\frac{\pi}{2}} |f'(x)|^2 dx\right)^2 - \left(\frac{\pi}{2}\int_0^{\frac{\pi}{2}} |f'(x)|^2 \cos 2x dx\right)^2\right\}^{1/2}. \tag{31}$$

如定理 4 的计算. 即可由(31)式导出(30)式.

参 考 文 献

[1] Z. Opial, *Sur une inegalite*, Ann. Polon. Math. 8(1960), 29—32.

[2] C. Olech, *A simple proof of a certain result of Z. Opial*, Ann. Polon. Math. 8(1960), 61—63

[3] L. K. Hua, On an inequality of Opial, *Sci. Sinica*, 14(1965), 789—790.

[4] Hu Ke, *On an inequality and its applications*, Sci. Sinica, Vol. X X Ⅳ, No. 8, 1047—1055.

[5] G. S. Yan, *On a certian result of Z. Opial*, Proc. Japan Acad, 42(1966), 78—83.

[6] 陈文忠、冯恭己、王兴华, Opial 不等式二十年, 2:4(1982), 151—166.

On Opial-Olech Inequality and Its Applications

Hu Ke

(Dept. of Math., Jiangxi Normal Univ., Nanchang)

Abstract

The main result of this paper is:

Theorem *Let $p \geq 0, q > 1, s = p/(p+q+1)$ and $f(x)$ be absolutely continuous on $[a,b]$ with $f(a)=f(b)=0$. Then*

$$\int_0^u |f|^p |f'|^q dx \leq \frac{1}{(p+q)^s}\left(\frac{n}{2}\right)^p \left(\int_0^h |f'|^{p+q} dx\right)$$

$$\left\{1 - \frac{1}{4}\left[\left(\int_0^h \cos\frac{2\pi x}{h}|f'|^{p+q}dx\right)/\left(\int_0^h |f'|^{p+q}dx\right]^2\right\}^{Q(p)}, \tag{A}$$

where $Q(p)=\frac{1}{2}s$ if $p+q>2, Q(p)=\frac{p}{2}$ if $1<p+q>2$.

(A) becomes Hua-Opial inequality if its right hand side is zero.