Proof The proof immediately follows from Theorem 3 and Corollary 2 and Corollary 3.

References

[1] P. Veselin, *Über halbprimäre ideale und halbprimäre ringe*, Mat. Vesnik, **13**(28)(1976), 327—329.

[2] Dong Xuedong, *On weakly primary rings (in Chinese)*, Northeastern Math. J., **4(3)**(1988), 316—322.

[3] Dong Xuedong, *On an open question of S. Singh and R. Kumar*, Chinese Science Bulletin, **35**(1990), 1503—1504.

[4] Dong Xuedong, *Commutative rings in which each nonzero ideal is a power of a prime ideal (in Chinese)*, Chinese Quarterly Journal of Math., **5**(1990), 109—114.

[5] Dong Xuedong, *Pseudo primary rings*, Northeastern Math. J., **7(4)**(1991), 448—452.

[6] J. R.Hedstrom, and E. G.Houston, *Pseudo-valuation domains*, Pacific J. Math., **75**(1978), 137—147.

[7] A. Okabe, *Some results on pseudo-valuation domains*, Tsukuba J. Math., **8(2)**(1984), 333-338.

[8] O. Zariski, and P. Samuel, *Commutative Algebra*, Vol. I, Van Nostrand, Princeton, N.J., 1958, 239—259.

关于半准素环

董学东

(辽宁师范大学数学系，大连 116022)

摘　要

交换环 R 称为(受限制的) 半准素环，如果对 R 的每个(非零) 主理想 A，都有 $\sqrt{A}$ 是 R 的素理想．本文刻画了受限制的半准素环，给出了有单位元的Noether 受限制的半准素环的分类以及半准素整环是伪赋值整环的一个条件

Journal of Mathematical Research & Exposition
Vol.14, No.3, Aug. 1994

On Half-Primary Rings *

Dong Xuedong
(Dept. of Math., Liaoning Normal University, Dalian,China)

Abstract A commutative ring R is called a (restricted) half- primary ring if for each (non-zero) principal ideal A of $R, \sqrt{A}$ is a prime ideal. In this paper we characterize the restricted half-primary rings, and give a complete classification of Noetherian restricted half-primary rings with identity and a condition that a half-primary domain is a pseudo-valuation domain.

Key words half-primary ring, pseudo-valuation domain, commutative ring.

A commutative ring R is called a (restricted) half-primary ring if for each (non-zero) principal ideal A of $R, \sqrt{A}$ is a prime ideal. Rings considered in [1]—[5] are all restricted half-primary rings. Pseudo-valuation domains defined in [6] and [7] are half-primary domains. In this paper we first characterize restricted half-primary rings and then give a complete classification of Noetherian restricted half-primary rings with identity and therefore generalize some results of [1], [2] and [5]. Finally we discuss relationship between half-primary domains and pseudo-valuation domains.

Throughout this paper all rings will be commutative but may not possess identity. Spec$R$ will denote the set of proper prime ideals of a ring R. An ideal A of a ring R is called half-primary if whenever $x, y \in R$ and $xy \in A$ there is a positive integer n such that $x^n \in A$ or $y^n \in A$[1]. A domain with identity R is called pseudo-valuation domain if whenever a prime ideal P contains the product xy of two elements of the quotient field of R then $x \in P$ or $y \in P$[6]. A ring R with identity is called a primary ring if $|\mathrm{Spec}R| = 1$[2].

Theorem 1 *The following conditions are equivalent for any ring R:*

(1) R is a restricted half-primary ring.

(2) The radical of each non-zero ideal of R is a prime ideal.

(3) Each non-zero ideal of R is half-primary.

(4) For each non-zero ideal A of R, $V(A) = \{P \in \mathrm{Spec}R \cup R \mid A \subseteq P\}$ is totally ordered.

Proof (1)$\longrightarrow$ (2): Let A be any non-zero ideal of R. Choose $0 \neq z \in A$, then $\sqrt{(z)}$ is a prime ideal and $\sqrt{(z)} \subseteq \sqrt{A}$. Let $x, y \in R, xy \in \sqrt{A}$ and $x \notin \sqrt{A}$. If $xy = 0$, then $xy \in \sqrt{(z)}$ and $x \notin \sqrt{(z)} \subseteq \sqrt{A}$ and therefore $y \in \sqrt{(z)} \subseteq \sqrt{A}$. If $xy \neq 0$, then

*Received Jun. 20, 1992. Project Supported by the Natural Science Foundation of Liaoning Province.

$\sqrt{(xy)}$ is a prime ideal. From $xy \in \sqrt{(xy)} \subseteq \sqrt{A}$ and $x \notin \sqrt{(xy)} \subseteq \sqrt{A}$ it follows that $y \in \sqrt{(xy)} \subseteq \sqrt{A}$, so $\sqrt{A}$ is a prime ideal.

(2)$\longrightarrow$ (3): Let A be any non-zero ideal of R. Then $\sqrt{A}$ is a prime ideal of R. If $x, y \in R, xy \in A \subseteq \sqrt{A}$, then $x \in \sqrt{A}$ or $y \in \sqrt{A}$. Hence there is a positive integer n or m such that $x^n \in A$ or $y^m \in A$. So A is half-primary.

(3)$\longrightarrow$ (4): For each non-zero ideal A of R, let $P_1, P_2 \in V(A) = \{P \in \mathrm{Spec}R \cup R | A \subseteq P\}$. If $P_1 \not\subseteq P_2$ and $P_2 \not\subseteq P_1$, choose $x \in P_1 - P_2$ and $y \in P_2 - P_1$, then $xy \in P_1 \cap P_2 \supseteq A \neq (0)$. So there exists a positive integer n such that $x^n \in P_1 \cap P_2$ or $y^n \in P_1 \cap P_2$ which in turn implies $x \in P_2$ or $y \in P_1$, a contradiction. Thus $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$ and therefore $V(A)$ is totally ordered.

(4)$\longrightarrow$ (1): Let $x \neq 0 \in R$. Then $V(x) = \{P \in \mathrm{Spec}R \cup R | P \supseteq (x)\}$ is totally ordered. So $\sqrt{(x)} = \bigcap_{P \in V(x)} P$ is a prime ideal. Thus R is a restricted half-primary ring.

The following Corollary 1 and Corollary 2 are immediate consequences of Theorem 1.

Corollary 1 *The following conditions are equivalent for any ring R:*

(1) R is a half-primary ring.

(2) The radical of each ideal of R is a prime ideal.

(3) Each ideal of R is half-primary.

(4) Spec$R$ is totally ordered.

Remark In Corollary 1 above, if R has identity, then (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4) were proved in [1].

Corollary 2 *A half-primary ring with identity is a local ring.*

Corollary 3 *A pseudo-valuation domain is a half-primary domain.*

Proof By Corollary 1.3 of [6], Spec$R$ is totally ordered in a pseudo- valuation domain R. So by Corollary 1, a pseudo-valuation domain is a half- primary domain.

Theorem 2 *Let R be a Noetherian ring with identity. Then R is a restricted half-primary ring if and only if R is a primary ring or R is a direct sum of two fields or R is a local ring with two proper prime ideals or R is a local ring with three proper prime ideals P, Q and M, where $P \cap Q = (0)$.*

Proof ($\Longrightarrow$): We first prove dim $R \leq 1$. If dim $R > 1$, then in R there is a strict ascending chain of proper prime ideals $P_1 \subset P_2 \subset P_3$. Choose $x \in P_2 - P_1$, $y \in P_3 - P_2$, then $xy \neq 0$ (otherwise, $xy = 0 \in P_1, x \notin P_1$ implies $y \in P_1 \subseteq P_2$ a contradiction). It is clear $(x, y) \subseteq P_2 \cap \sqrt{(y)}$. By Theorem 1, $P_2 \subseteq \sqrt{(y)}$, or $\sqrt{(y)} \subseteq P_2$. But since $y \notin P_2, P_2 \subset \sqrt{(y)}$. So $P_1 \subset P_2 \subset \sqrt{(y)}$ which contradicts Krull's Principal Ideal Theorem [8]. Thus dim $R \leq 1$.

If dim $R = 0$, then R is a primary ring when $|\mathrm{Spec}R| = 1$. When $|\mathrm{Spec}R| > 1$, choose $P, Q \in \mathrm{Spec}R$, then by Theorem 1, $P \cap Q = (0)$. Obviously, $P + Q = R$. So by Cinese Remainder Theorem, $R \cong R/P \oplus R/Q$, that is, R is a direct sum of two fields when $|\mathrm{Spec}R| > 1$.

If dim $R = 1$, then in R there is a strict ascending chain of proper prime ideals

$P \subset M. M \in \text{Max } R$ is obvious. If $N \in \text{Max } R$, then $M \cap N \neq (0)$ and therefore $M \subseteq N$ or $N \subseteq M$ by Theorem 1. So $M = N$. This shows R is a local ring. If $Q \in \text{Spec}R$ and $Q \neq M$, then $Q \cap P \neq (0)$ or $Q \cap P = (0)$. When $Q \cap P \neq (0), Q \subseteq P$ or $P \subseteq Q$ by Theorem 1. But since $\dim R = 1, P = Q$. So R is a local ring with two proper prime ideals when $Q \cap P \neq (0)$. When $Q \cap P = (0)$, for any $P_1 \in \text{Spec}R$, $QP \subseteq Q \cap P = (0) \subseteq P_1$ and hence $Q \subseteq P_1 \subseteq M$ or $P \subseteq P_1 \subseteq M$. So $P_1 = P$ or $P_1 = Q$ or $P_1 = M$ since $\dim R = 1$. Thus R is a local ring with three proper prime ideals P, Q and M, when $P \cap Q = (0)$.

($\Longleftarrow$): If R is a primary ring or R is a direct sum of two fields, then by Theorem 1 of [2], R is a restricted weakly primary ring and hence a restricted half-primary ring. If R is a local ring with two proper prime ideals or R is a local ring with three proper prime ideals P, Q and M, where $P \cap Q = (0)$, then it is easily proved that for each non-zero ideal A of R, $|V(A)| \leq 3$ and $V(A)$ is totally ordered. By Theorem 1, R is a restricted half-primary ring.

Now we discuss relationship between the half-primary domains and the pseudo- valuation domains. By Corollary 3, the pseudo-valuation domains are the half-primary domains. We wonder whether each half-primary domain is a pseudo- valuation domain. The following Example shows that even a weakly primary domain need not be any pseudo-valuation domain.

Example Let k be a field and $R = F + x^2F[[x]]$ be a subring of the formal power series ring $F[[x]]$. Then since a element of R is a unit if and only if its constant term is a unit of $F, M = x^2F[[x]]$ is the maximal ideal of R. It is obvious that $F[[x]]$ is an integral extension ring of R and $F[[x]]$ has the unique non-zero prime ideal $xF[[x]]$. By going-up Theorem ([8]), $M = x^2F[[x]]$ is the only non-zero prime ideal of R. So R is a weakly primary domain and therefore a half-primary domain. It is ease to see that x is in the quotient field of R. But from $x^2 \in M$ it does not follow that $x \in M$. So R is not a pseudo-valuation domain.

Theorem 3 *A domain R is a pseudo-valuation domain if and only if R is a local ring and whenever $x, y \in R$ with $x \nmid y$ and $y \nmid x$ we have $((x) : y) = ((y) : x) = M$, where $M \in$ Max R.*

Proof ($\Longrightarrow$): By Corollary 2, R is a local ring. Let $M \in \text{Max } R$. If $x, y \in R$ with $x \nmid y$ and $y \nmid x$, then for any $r \in ((x) : y)$, we have $ry \in (x)$ and hence r is not any unit of R. So $r \in M$. This shows $((x) : y) \subseteq M$. Likewise $((y) : x) \subseteq M$. On the other hand, by Proposition 1.2 of [6], $(x/y)M \subseteq M$ and $(y/x)M \subseteq M$ which imply $M \subseteq ((x) : y) \cap ((y) : x)$. So $((x) : y) = ((y) : x) = M$.

($\Longleftarrow$): Let $c = (x/y) \in K - R$, where K is the quotient field of R and $x, y \in R$. Then $y \nmid x$. If $c^{-1} = (y/x) \in R$, then $c^{-1}M \subseteq M$, where $M \in \text{Max } R$. If $c^{-1} \notin R$, then $x \nmid y$. So $((x) : y) = ((y) : x) = M$. It is easily proved that $R \cap cR = M$ and $R \cap c^{-1}R = M$. Hence (multiplying the first equation by c^{-1}), $c^{-1}M = c^{-1}R \cap R = M$. By Theorem 1.4 and Proposition 1.2 of [6], R is a pseudo- valuation domain.

Corollary A domain R is a pseudo-valuation domain if and only if R is a half-primary domain and whenever $x, y \in R$ with $x \nmid y$ and $y \nmid x$ we have $((x) : y) = ((y) : x) = M$, where $M \in \text{Max } R$.

Proof The proof immediately follows from Theorem 3 and Corollary 2 and Corollary 3.

References

[1] P. Veselin, *Über halbprimäre ideale und halbprimäre ringe*, Mat. Vesnik, **13**(28)(1976), 327—329.

[2] Dong Xuedong, *On weakly primary rings (in Chinese)*, Northeastern Math. J., **4(3)**(1988), 316—322.

[3] Dong Xuedong, *On an open question of S. Singh and R. Kumar*, Chinese Science Bulletin, **35**(1990), 1503—1504.

[4] Dong Xuedong, *Commutative rings in which each nonzero ideal is a power of a prime ideal (in Chinese)*, Chinese Quarterly Journal of Math., **5**(1990), 109—114.

[5] Dong Xuedong, *Pseudo primary rings*, Northeastern Math. J., **7(4)**(1991), 448—452.

[6] J. R.Hedstrom, and E. G.Houston, *Pseudo-valuation domains*, Pacific J. Math., **75**(1978), 137—147.

[7] A. Okabe, *Some results on pseudo-valuation domains*, Tsukuba J. Math., **8(2)**(1984), 333-338.

[8] O. Zariski, and P. Samuel, *Commutative Algebra*, Vol. I, Van Nostrand, Princeton, N.J., 1958, 239—259.

关于半准素环

董学东

(辽宁师范大学数学系，大连 116022)

摘 要

交换环R称为(受限制的)半准素环，如果对R的每个(非零)主理想A，都有$\sqrt{A}$是R的素理想. 本文刻画了受限制的半准素环，给出了有单位元的Noether受限制的半准素环的分类以及半准素整环是伪赋值整环的一个条件