Journal of Mathematical Research & Exposition
Vol.14, No.4, Nov. 1994

A Formula Involving Convolutions and Partition- Sums*

Leetsch C. Hsu
(Inst. of Math. Sci., Dalian Univ. of Tech., Dalian 116024)

Here we give a fruitful extension of a remarkable result due to T.H. Savits and G.M. Constantine (cf. Tech. Report, No. 92-02, 1992, University of Pittsburgh). Let $\sigma(n)$ and $\sigma(n,k)$ denote respectively the set of partitions of n and the subset containing partitions of n with k-parts, viz.

$$\sigma(n,k) := \left\{1^{r_1}2^{r_2}\cdots n^{r_n} \,|r_1+2r_2+\cdots+nr_n=n, \quad \sum r_i=k, \quad r_i\geq 0\right\}$$

Let $f(r)$ be defined on $\{0,1,\cdots,n\}$ with $f(0)=1$. Denote

$$S(f,k,n) = \sum_{r_1+\cdots+r_k=n} f(r_1)\cdots f(r_k) \tag{1}$$

$$T(f,k,n) = \sum_{\sigma(n,k)} f(1)^{r_1}\cdots f(n)^{r_n}/r_1!\cdots r_n! \tag{2}$$

where (1) stands for the k-fold convolution of f (with $r_i\geq 0$), and the RHS of (2) is taken over all the partitions $1^{r_1}\cdots n^{r_n}$ of n with k-parts. Then one can prove the following theorem with the aid of Faadi Bruno's formula.

Theorem Given a formal power series

$$G(t)=\sum_{k=0}^{\infty} g(k)t^k \tag{3}$$

with complex coefficients $g(k)'s$. Then there holds formally

$$\sum_{k=1}^{\infty} g(k)t^k S(f,k,n) = \sum_{k=1}^{n} G^{(k)}(t)t^k T(f,k,n) \tag{4}$$

where $G^{(k)}(t)$ is the k-th order formal derivative of $G(t)$.

Simple examples may be readily given as follows

(i) $$\sum_{k=1}^{\infty} t^k S(f,k,n) = \sum_{k=1}^{n} \frac{k!}{1-t}\left(\frac{t}{1-t}\right)^k T(f,k,n), \quad (|t|<1)$$

(ii) $$\sum_{k=1}^{\infty} g(k)t^k \binom{kx}{n} = \sum_{k=1}^{n} G^{(k)}(t)t^k \sum_{\sigma(n,k)} \binom{x}{1}^{r_1}\cdots\binom{x}{n}^{r_n}/r_1!\cdots r_n!$$

Further extensions are being searched jointly with Constantine and Savits.

*Received May 5, 1994.