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Certain Integral Operators for Starlike Functions *

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Abstract Some results concerning containment of various families of starlike functions defined on certain integral operators are obtained.

Keywords starlike function, integral operators.

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1. Introduction and definitions

Let $A(n)$ denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad n \in N = \{1,2,3,\cdots\} \tag{1}$$

regular in the unit disk $\Delta = \{z : |z| < 1\}$.

A function $f(z)$ belonging to the class $A(n)$ is said to be in the class $S^*(n)$ if and only if

$$\operatorname{Re}\{\frac{zf'(z)}{f(z)}\} > 0 \quad (z \in \Delta).$$

Further, a function $f(z)$ belonging to the class $A(n)$ is said to be in the class $K(n)$ if and only if

$$\operatorname{Re}\{1 + \frac{zf''(z)}{f'(z)}\} > 0 \quad (z \in \Delta).$$

It is well known that $f \in K(n)$ if and only if $zf'(z) \in S^*(n)$ and $K(n) \subset S^*(n)$.

For any real number α let the operator I^α operating on $f \in A(n)$ be defined by

$$I^\alpha f(z) = z + \sum_{k=n+1}^{\infty} (\frac{k+1}{2})^{-\alpha} a_k z^k.$$

A function $f(z)$ given by (1) is said to be in the class $S^*(n,\alpha)$ if $I^\alpha f(z) \in S^*(n)$ and is said to be in the class $K(n,\alpha)$ if $I^\alpha f(z) \in K(n)$.

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In this paper we investigate certain properties of the classes $S^*(n,\alpha)$ and $K(n,\alpha)$. Methods used here are similar to those in [1].

2. The classes $S^*(n,\alpha)$ and $K(n,\alpha)$

We need the following result due to Miller and Mocanu [2].

Lemma *Let $\phi(u,v)$ be a complex valued function, $\phi : D \to C, D \subset C \times C$ (C the complex plane) and let $u = u_1 + iu_2, v = v_1 + iv_2$. Suppose that the function $\phi(u,v)$ satisfies the following conditions:*

(i) $\phi(u,v)$ is continuous in D;

(ii) $(1,0) \in D$ and $\text{Re}\ \{\phi(1,0)\} > 0$;

(iii) for all $(iu_2, v_1) \in D$ and such that $v_1 \le -n(1+u_2^2)/2$, $\text{Re}\{\phi(iu_2, v_1)\} \le 0$.

Let $p(z) = 1 + p_n z^n + p_{n+1}z^{n+1} + \cdots$ be regular in the unit disk such that $(p(z), zp'(z)) \in D$ for all $z \in \Delta$. If $\text{Re}\{\phi(p(z), zp'(z))\} > 0 (z \in \Delta)$, then $\text{Re}\{p(z)\} > 0 (z \in \Delta)$.

Using the above lemma we prove.

Theorem 1 *For any real number $\alpha, S^*(n,\alpha) \subset S^*(n,\alpha+1)$.*

Proof Let $f(z) \in S^*(n,\alpha)$. Define the function $p(z)$ by

$$\frac{z(I^{\alpha+1}f(z))'}{I^{\alpha+1}f(z)} = p(z) \tag{2}$$

Then $p(z) = I + p_n z^n + p_{n+1}z^{n+1} + \cdots$ is regular in Δ. So, using the identity (easy to verify)

$$z(I^{\alpha+1}f(z))' = 2I^\alpha f(z) - I^{\alpha+1}f(z)$$

(2) may be written as

$$\frac{I^\alpha f(z)}{I^{\alpha+1}f(z)} = \frac{1}{2}(1+p(z)). \tag{3}$$

Differentiating (3) logarithmically we obtain

$$\frac{z(I^\alpha f(z))'}{I^\alpha f(z)} = p(z) + \frac{zp'(z)}{1+p(z)}$$

or

$$\text{Re}\{\frac{z(I^\alpha f(z))'}{I^\alpha f(z)}\} = \text{Re}\{p(z) + \frac{zp'(z)}{1+p(z)}\} > 0.$$

Let $p(z) = u = u_1 + iu_2, zp'(z) = v = v_1 + iv_2$, and $\phi(u,v) = u + \dfrac{v}{1+u}$. Then

(i) $\phi(u,v)$ is continuous in $D = (C - \{-1\} \times C)$;

(ii) $(1,0) \in D$ and $\text{Re}\{\phi(1,0)\} = 1 > 0$;

(iii) for all $(iu_2, v_1) \in D$ and such that $v_1 \le -n(1+u_2^2)/2$,

$$\text{Re}\{\phi(iu_2, v_1)\} = \frac{v_1}{1+u_2^2} \le -\frac{n(1+u_2^2)}{2(1+u_2^2)} \le -\frac{n}{2} \le 0.$$

Thus the funtion $\phi(u,v)$ satisfies the conditions of the above Lemma. It follows that $\mathrm{Re}\{p(z)\} > 0$. That is $\mathrm{Re}\{\frac{z(I^{\alpha+1}f(z))'}{I^{\alpha+1}f(z)}\} > 0$ $(z \in \Delta)$. Hence $f \in S^*(n, \alpha+1)$.

Theorem 2 *For any real number* α, $K(n,\alpha) \subset K(n,\alpha+1)$

Proof Observe that

$$\begin{aligned} f \in K(n,\alpha) \quad &\Leftrightarrow \quad I^\alpha f(z) \in K(n) \Leftrightarrow z(I^\alpha f(z))' \in S^*(n) \Leftrightarrow I^\alpha(zf'(z)) \in S^*(n) \\ &\Leftrightarrow \quad zf'(z) \in S^*(n,\alpha) \Rightarrow zf'(z) \in S^*(n,\alpha+1) \Leftrightarrow I^{\alpha+1}(zf'(z)) \in S^*(n) \\ &\Leftrightarrow \quad z(I^{\alpha+1}f(z))' \in S^*(n) \Leftrightarrow I^{\alpha+1}f(z) \in K(n) \Leftrightarrow f(z) \in K(n,\alpha+1). \end{aligned}$$

Theorem 3 *If the function $f(z)$ defined by (1) is in the class $S^*(n,\alpha)$, then*

$$\mathrm{Re}\{(\frac{I^\alpha f(z)}{z})^{\beta-1}\} > \frac{n}{2(\beta-1)+n} \quad (z \in \Delta)$$

where $1 < \beta \le (n+2)/2$.

Corollary If the function $f(z)$ defined by (1) is in the class $K(n,\alpha)$, then

$$\mathrm{Re}\{(I^\alpha f(z))'\}^{\beta-1} > \frac{n}{2(\beta-1)+n} \quad (z \in \Delta)$$

where $1 < \beta \le (n+2)/2$. Proofs of Theorem 3 and its corollary are similar to those of analogous results in [1].

References

[1] N.E.Cho, O.Kwon and S.Owa, *Notes on the Ruscheweyh derivatives*, Bull, Cal. Math. Soc., 83(1991), 35-39.

[2] S.S.Miller and P.T.Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl., 65(1978), 289-305.