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The $\omega_{\alpha+1}$-Compact T_1-Space with Submeta-$\mathcal{B}$ -Property is ω_α-Lindelöf *

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Abstract The main result of the paper is that the $\omega_{\alpha+1}$-compact T_1-space with submeta-$\mathcal{B}$-property is ω_α-Lindelöf. This result improves the main results of [1].

Key words ω_α-compact, ω_α-Lindelöf, submeta-$\mathcal{B}$-property.

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In this paper, the space means the topological space without any separation axioms assumed unless especially stated. $\omega = \omega_0$ denotes the first infinite ordinal. For any ordinal $\alpha > 0$, ω_α denotes the α-th uncountable ordinal. The cardinal of a set A is denoted by $|A|$. Cardinals are initial ordinals. The space X is said to have $\mathcal{B}$- property[1] if for any monotone increasing open cover $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ of X, there is a monotone increasing open cover $\mathcal{V} = \{V_\alpha : \alpha \in A\}$ of X s.t. $\bar{V}_\alpha \subset U_\alpha$ for any $\alpha \in A$. $\mathcal{B}$-property is between paracompactness and countable paracompactness and is studied by many authors[2]. For the sake of unity, we appoint that $|A| \le \omega_{-1}$ denotes $|A|$ is a finite cardinal and ω_{-1}-Lindelöf denotes compact. After making these appointments, all results in the paper hold for $\alpha \ge -1$ unless especially stated.

Definition 1 *A space X is called ω_α-Lindelöf if any open cover $\mathcal{U}$ of X has a subcover $\mathcal{V}$ s.t. $|\mathcal{V}| \le \omega_\alpha$.*

Definition 2 *A space X is called ω_α-compact if any subset B with the cardinal ω_α has an accumulation point.*

Clearly, the ω_{-1}-Lindelöf (ω_0-Lindelöf) space coincides with the compact (Lindelöf) space, and if X is T_1, then X is ω_0-compact iff X is countably compact. The following implications are obvious:

$$\begin{array}{ccccccccccccc}
\text{com.} & \to & \omega_0\text{-Lin.} & \to & \omega_1\text{-Lin.} & \to\cdots\to & \omega_\alpha\text{-Lin.} & \to & \omega_{\alpha+1}\text{-Lin.} & \to\cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\omega_0\text{-com.} & \to & \omega_1\text{-com.} & \to & \omega_2\text{-com.} & \to\cdots\to & \omega_{\alpha+1}\text{-com.} & \to & \omega_{\alpha+1}\text{-Lin.} & \to\cdots
\end{array}$$

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where com. = compact, Lin. = Lindelöf.

None of the above implications is reversible:

Example (1) $[0,\omega_{\alpha+2})$ is $\omega_{\alpha+1}$-compact, but it is not ω_α-Lindelöf: Let $A \subset [0,\omega_{\alpha+2})$ and $|A| = \omega_{\alpha+1}$, then $\beta_0 = \sup A < \omega_{\alpha+2}$ since $\omega_{\alpha+2}$ is regular. Since $[0,\beta_0]$ is compact the infinite set $B = [0,\beta_0] \cap A$ has an accumulation point $\xi \in [0,\beta_0]$ which is also an accumulation point of A. Thus $[0,\omega_{\alpha+2})$ is $\omega_{\alpha+1}$-compact. Take an open cover $\mathcal{U} = \{[0,\beta] : \beta \in [0,\omega_{\alpha+2})\}$ of X. If $\mathcal{U}' \subset \mathcal{U}$ and $|\mathcal{U}'| \leq \omega_\alpha$, then $\mathcal{U}'$ can not cover $[0,\omega_{\alpha+2})$. Therefore $[0,\omega_{\alpha+2})$ is not ω_α-Lindelöf. (2) Let X be a discrete space and $|X| = \omega_{\alpha+1}$. Then X is an $\omega_{\alpha+1}$-Lindelöf ($\omega_{\alpha+2}$-compact) space, but it is not an ω_α-Lindelöf ($\omega_{\alpha+1}$-compact) space.

A question is naturally asked: under what condition the $\omega_{\alpha+1}$- compactness implies the ω_α-Lindelöfness? Our Theorem answers this question.

Definition 3 *The space X is said to have submeta-$\mathcal{B}$-property if every infinite open cover $\mathcal{U}$ of X has an open refinement sequence $\{\mathcal{V}_n : n \in \omega\}$ s.t. for every $x \in X$, there is an $n(x) \in \omega$ s.t. $|\{V \in \mathcal{V}_{n(x)} : x \in V\}| < |\mathcal{U}|$.*

From the following Lemma 1, we can easily see that the $\mathcal{B}$-property implies the submeta-$\mathcal{B}$-property. But the implication is not reversible: Let F be Bing's Example G[2], then the subspace Y of F decribed in [3] is metacompact and so Y has submeta-$\mathcal{B}$-property, but Y does not have $\mathcal{B}$-property (cf. [2] and [3]).

The sequence $\{\mathcal{V}_n : n \in \omega\}$ of open covers of the space X is said to be an open point star refinement sequence of the open cover $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ if for every $x \in X$, there exist an $n(x) \in \omega$ and an $\alpha(x) < \kappa$ s.t. $\mathrm{st}(x, \mathcal{V}_{n(x)}) \subset U_{\alpha(x)}$.

Lemma 1 *For a space X, the following are equivalent:*

(1) X has submeta-$\mathcal{B}$-property.

(2) Any monotone increasing open cover $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ of X has an open point star refinement sequence.

(3) Any monotone increasing open cover $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ of X has a closed cover $\mathcal{F} = \{F_{n\alpha} : n \in \omega, \ \alpha < \kappa\}$ s.t. $F_{n\alpha} \subset U_\alpha$ and $F_{n\alpha_1} \subset F_{n\alpha_2}$ if $\alpha_1 < \alpha_2$.

Proof (1)→ (2): If $\mathrm{cf}\kappa = \kappa$, then by (1) $\mathcal{U}$ has an open refinement sequence $\{\mathcal{V}_n : n \in \omega\}$ s.t. for every $x \in X$, there is an $n(x) \in \omega$ and $|\{V \in \mathcal{V}_{n(x)} : x \in V\}| < \kappa$. Let $\mathcal{V}' = \{V \in \mathcal{V}_{n(x)} : x \in V\}$. Since $\mathcal{V}_{n(x)}$ is a refinement of $\mathcal{U}$, for every $V \in \mathcal{V}'$, there is an $\alpha(V) < \kappa$ s.t. $V \subset U_{\alpha(V)}$. Since $\mathrm{cf}\kappa = \kappa$ and $|\mathcal{V}'| < \kappa$, there is an $\alpha(x) < \kappa$ s.t. for every $V \in \mathcal{V}'$, $\alpha(V) < \alpha(x)$. Therefore $\mathrm{st}(x, \mathcal{V}_{n(x)}) \subset U_{\alpha(x)}$. So $\{\mathcal{V}_n : n \in \omega\}$ is an open point star refinement sequence of $\mathcal{U}$. If $\mathrm{cf}\kappa < \kappa$, then for $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$, κ has a monotone increasing cofinal subset $\{\alpha_\eta : \eta < \mathrm{cf}\kappa\}$. Put $V_\eta = U_{\alpha_\eta}$. Then $\mathcal{V} = \{V_\eta : \eta < \mathrm{cf}\kappa\}$ is a monotone increasing open cover and $|\mathcal{V}| = \mathrm{cf}\kappa$ is regular. According to the above proof $\mathcal{V}$ has an open point star refinement sequence, so does $\mathcal{U}$.

(2) → (3): By (2), $\mathcal{U}$ has an open point star refinement sequence, $\{\mathcal{V}_n : n \in \omega\}$. Put $F_{n\alpha} = \overline{\{x \in U_\alpha : \mathrm{st}(x, \mathcal{V}_n) \subset U_\alpha\}}$, then $\{F_{n\alpha} : n \in \omega, \ \alpha < \kappa\}$ is a closed cover of X s.t. $F_{n\alpha} \subset U_\alpha$ and if $\alpha_1 < \alpha_2$, then $F_{n\alpha_1} \subset F_{n\alpha_2}$.

(3) → (1). Let $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ be an infinite open cover of X. Put $V_\alpha = \bigcup_{\beta<\alpha} U_\beta$, then

$\mathcal{V} = \{V_\alpha : \alpha < \kappa\}$ is a monotone increasing open cover of X, by (3), there is a closed cover $\{F_{n\alpha} : n \in \omega,\ \alpha < \kappa\}$ s.t. $F_{n\alpha} \subset V_\alpha$, and when $\alpha_1 < \alpha_2$, $F_{n\alpha_1} \subset F_{n\alpha_2}$. Put $V_{n\alpha} = U_\alpha - F_{n\alpha}$, $n \in \omega, \alpha < \kappa$, and $\mathcal{V}_n = \{V_{n\alpha} : \alpha < \kappa\}$, $n \in \omega$. It is obvious that $\{\mathcal{V}_n : n \in \omega\}$ is an open refinement sequence of $\mathcal{U}$. For every $x \in X = \bigcup_{n\in\omega}\bigcup_{\alpha<\kappa} F_{n\alpha}$, there is the smallest $n(x)$ s.t. $x \in \bigcup_{\alpha<\kappa} F_{n(x)\alpha}$, and there is the smallest $\alpha(x)$ s.t. $x \in F_{n(x)\alpha(x)}$. If $\alpha \geq \alpha(x)+1$, then $x \in F_{n(x)\alpha}$ and so $x \notin U_\alpha - F_{n(x)\alpha} = V_{n(x)\alpha}$, therefore $|\{V \in \mathcal{V}_{n(x)} : x \in V\}| < \mathcal{V}$. This shows (1).

A space X is said to have property (*) if any monotone increasing open cover $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ of X has a closed refinement $\mathcal{F} = \{F_{n\alpha} : \alpha \in A, n \in \omega\}$ satisfying $F_{n\alpha} \subset U_\alpha$ for any $n \in \omega, \alpha \in A$.

Lemma 2 *Let X be a space, $\mathcal{A} = \{\mathcal{U} : \mathcal{U}$ is an open cover of X satisfying that if $\mathcal{V} \subset \mathcal{U}$ and $|\mathcal{V}| \leq \omega_\alpha$, then $\mathcal{V}$ does not cover $X\}$ and $\kappa = \min\{|\mathcal{U}| : \mathcal{U} \in \mathcal{A}\}$. If $\alpha = -1$, then κ is regular. If $\alpha \geq 0$ and X has property (*), then κ is also regular.*

Proof Suppose $\mathrm{cf}\kappa = \kappa$ and choose a $\mathcal{U} \in \mathcal{A}$ s.t. $\kappa = |\mathcal{U}|$. Let $f : \mathrm{cf}\kappa \to \kappa$ be a monotone increasing cofinal mapping, $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ and $W_\alpha = \bigcup_{\beta<\alpha} U_\beta$, $\alpha < \kappa$. Then $\mathcal{W} = \{W_{f(\alpha)} : \alpha < \mathrm{cf}\kappa\}$ is a monotone increasing open cover of X. Since $\mathrm{cf}\kappa < \kappa$, there is a $\kappa_1 \leq \omega_\alpha$ s.t. $\mathcal{W}' = \{W_{f(\alpha_\beta)} : \beta < \kappa_1\} \subset \mathcal{W}$ also covers X. Without loss of generality, we may assume that if $\beta_1 < \beta_2$, then $\alpha_{\beta_1} < \alpha_{\beta_2}$. If $\alpha = -1$, then κ_1 is finite. So $X = W_{f(\alpha_{\kappa_1-1})} = \bigcup_{\xi<f(\alpha_{\kappa_1-1})} U_\xi$. Since $f(\alpha_{\kappa_1-1}) < \kappa$, there is a finite set $\{\xi_1, \xi_2, \cdots, \xi_m\} \subset [0, f(\alpha_{\kappa-1}))$ s.t. $\{U_{\xi_1}, U_{\xi_2}, \cdots, U_{\xi_m}\} \subset \mathcal{U}$ covers X, this contradicts the hypothesis. If $\alpha \geq 0$ and X has property (*), then for $\mathcal{W}'$, there is a closed cover $\mathcal{F} = \{F_{n\beta} : \beta < \kappa_1, n \in \omega\}$ of X s.t. $F_{n\beta} \subset W_{f(\alpha_\beta)} = \bigcup_{\xi<f(\alpha_\beta)} U_\xi$ for any $\beta < \kappa_1$, $n \in \omega$. For every $n \in \omega$, the family $\{U_\xi : \xi < f(\alpha_\beta)\} \cup \{X - F_{n\beta}\}$ covers X and has the cardinal $< \kappa$. So it has a subfamily with the cardinal $\leq \omega_\alpha$ covering X. Thus the cover $\{U_\xi : \xi < f(\alpha_\beta)\}$ of $F_{n\beta}$ has a subcover $\mathcal{U}_{n\beta}$ with the cardinal $\leq \omega_\alpha$. Put $\mathcal{U}_\beta = \cup\{\mathcal{U}_{n\beta} : n \in \omega\}$, then $\mathcal{U}_\beta$ covers $F_\beta = \bigcup_{n\in\omega} F_{n\beta}$ and $|\mathcal{U}_\beta| \leq \omega_\alpha$. Therefore the subfamily $\mathcal{U}' = \cup\{\mathcal{U}_\beta : \beta < \kappa_1\}$ of $\mathcal{U}$ covers X since $\mathcal{F}$ covers X. But $|\mathcal{U}'| \leq \omega_\alpha$ and this contradicts the hypothesis. Therefore $\mathrm{cf}\kappa = \kappa$.

Theorem 1 *If a T_1-space X is $\omega_{\alpha+1}$-compact and has submeta-$\mathcal{B}$-property, then X is ω_α-Lindelöf.*

Proof Suppose X is not ω_α-Lindelöf. Let $\mathcal{A} = \{\mathcal{U} : \mathcal{U}$ is an open cover of X whose any subfamily with the cardinal $\leq \omega_\alpha$ can not cover $X\}$ and $\kappa = \min\{|\mathcal{U}| : \mathcal{U} \in \mathcal{A}\}$. Take a $\mathcal{U} \in \mathcal{A}$ s.t. $|\mathcal{U}| = \kappa$. If $\alpha \geq 0$, then $\kappa \geq \omega_{\alpha+1}$. If $\alpha = -1$, then $\kappa \geq \omega_1$ because if $\kappa = \omega_0$ then $\mathcal{U}$ has a finite subcover since the ω_0-compact and T_1 space is countably compact. By Lemma 1, submeta-$\mathcal{B}$-property implies property (*). According to Lemma 2, $\mathrm{cf}\kappa = \kappa$. We may assume that $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ satisfies that for any $\alpha < \kappa$, $U_\alpha - \bigcup_{\beta<\alpha} U_\beta \neq \emptyset$. Let

$\{\alpha_\eta : \eta < \kappa\}$ be a monotone increasing cofinal subset of κ and $V_\eta = \bigcup_{\alpha < \alpha_\eta} U_\alpha$. By Lemma 1 the monotone increasing open cover $\mathcal{V} = \{V_\eta : \eta < \kappa\}$ of X has an open point star refinement sequence $\{\mathcal{V}_n : n \in \omega\}$. Take an $x_0 \in X$, then there exist an $n(x_0) \in \omega$ and an $\eta_0 < \kappa$ s.t. $\mathrm{st}(x_0, \mathcal{V}_{n(x_0)}) \subset V_{\eta_0}$. Take an $x_1 \in X - V_{\eta_0}$, then there exist an $n(x_1) \in \omega$ and an $\eta_1 < \kappa$ s.t. $\mathrm{st}(x_1, \mathcal{V}_{n(x_1)}) \subset V_{\eta_1}$. Suppose for υ, when $\rho < \upsilon$, $x_\rho, n(x_\rho)$ and η_ρ have been defined. If $\xi = \sup\{\eta_\rho : \rho < \upsilon\} < \kappa$, take an $x_\upsilon \in X - V_\xi$, then there exist an $n(x_\upsilon)$ and an η_υ s.t. $\mathrm{st}(x_\upsilon, \mathcal{V}_{n(x_\upsilon)}) \subset V_{\eta_\upsilon}$. If $\xi = \kappa$, then $\upsilon = \kappa$ and we finish the definition. Put $B = \{x_\rho : \rho < \kappa\}$. Obviously, if $\rho_1 < \rho_2$, then $\eta_{\rho_1} < \eta_{\rho_2}$. There must be $n_0 \in \omega$ and $A \subset B$ s.t. $|A| = \kappa$ and for every $x_{\rho_\lambda} \in A$, $\mathrm{st}(x_{\rho_\lambda}, \mathcal{V}_{n_0}) \subset \mathcal{V}_{\eta_{\rho_\lambda}}$. We may assume that $A = \{x_{\rho_\lambda} : \lambda < \kappa\}$ satisfies $\rho_{\lambda_1} < \rho_{\lambda_2}$ if $\lambda_1 < \lambda_2$. For any $x \in X$, if $x \notin st(A, \mathcal{V}_{n_0})$, then there is a $V \in \mathcal{V}_{n_0}$ s.t. $x \in V$ and $V \cap A = \emptyset$. If $x \in \mathrm{st}(A, \mathcal{V}_{n_0})$, then there is an $x_{\rho_{\lambda_0}} \in A$ s.t. $x \in \mathrm{st}(x_{\rho_{\lambda_0}}, \mathcal{V}_{n_0}) \subset V_{\eta_{\rho_{\lambda_0}}}$. Clearly $\mathrm{st}(x_{\rho_{\lambda_0}}, \mathcal{V}_{n_0}) \cap A = \{x_{\rho_{\lambda_0}}\}$. thus A has no accumulation point since X is T_1. But $|A| = \kappa$. This contradicts $\omega_{\alpha+1}$-compactness.

Noticing the cases $\alpha = 0$ and $\alpha = -1$ in Theorem 1, we obtain

Corollary 1 *The regular T_1-space X is Lindelöf iff X is ω_1-compact and has submeta-$\mathcal{B}$-property.*

Corollary 2 *The T_2-space X is compact iff X is countably compact and has submeta-$\mathcal{B}$-property.*

Remark (1) By Lemma 1 the developable space has submeta-$\mathcal{B}$- property, so the developable T_1-space with $\omega_{\alpha+1}$-compactness is ω_α-Lindelöf. Thus if the T_1-space X has submeta-$\mathcal{B}$- property (or X is developable), then ω_α-Lindelöfness and $\omega_{\alpha+1}$-compactness are equivalent. (2) Corollary 1 and Corollary 2 improve the main results of [1], i.e., a regular T_1-space X is Lindelöf (compact) iff X is ω_1-compact (countably compact) and has $\mathcal{B}$-property.

References

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具有次亚$\mathcal{B}$性质的$\omega_{\alpha+1}$-紧T_1空间是ω_α-Lindelöf空间

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摘　要

在本文中，我们证明了具有次亚$\mathcal{B}$性质的$\omega_{\alpha+1}$-紧T_1空间是ω_α-Lindelöf空间. 此结果改进并推广了[1]中的主要结果.