Journal of Mathematical Research & Exposition
Vol.15, No.1, 021-024, Feb. 1995

G-Semilocal Rings and Homological Dimensions *

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Abstract We discuss an extended class of the Semilocal rings which are called G-semilocal rings. For any G-semilocal ring R, the homological dimension of R has been derived by using the residue classes modulo $\mathrm{Soc}(_R R)$ and $J(R)\cap \mathrm{Soc}(_R R)$ respectively.

Keywords semilocal ring, homological dimension, Jacobson radical

Classification AMS(1991) 13D05, 16E10/CCL O153.3

1. G-Semilocal Rings

Throughout this paper, R always denotes an associative ring with an identity, and all modules are unital. For any ring R, S will denote $\mathrm{Soc}(_R R)$ and J the Jacobson radical of R.

Let R be a ring, if $(I_\alpha)_{\alpha\in A}$ is the set of all the minimal left ideal of R, then $S = \sum_{\alpha\in A} I_\alpha$. By [3,Lemma 9.2] there is a subset $B \subset A$ such that

$$S = (S\bigcap J)\bigoplus(\bigoplus_{\beta\in B} I_\beta), \tag{1}$$

clearly $I_\beta^2 = I_\beta$. If B is finite, then

$$S = (S\bigcap J)\bigoplus(\bigoplus_{i=1}^{n} I_i), \tag{2}$$

where $I_i^2 = I_i \in (I_\alpha)_{\alpha\in A}$.

Definition 1.1 *A ring R is G-semilocal if there are minimal left ideals $I_1, I_2, \cdots, I_n$ such that $S = (S\bigcap J)\bigoplus(\bigoplus_{i=1}^{n} I_i)$, clearly $I_i^2 = I_i$.*

Corollary 1.2 *Let R be a ring, if all but a finite number of minimal left ideals of R are nilpotents, then R is a G-semilocal ring.*

Proposition 1.3 *Let R be a semilocal ring, then R is G- semilocal.*

*Received Oct.17, 1992.

Proof Let $\overline{R} = R/S \cap J$. We have that $S = (S \cap J) \oplus (\bigoplus_{\beta \in B} I_\beta)$ and $I_\beta^2 = I_\beta$, thus $\overline{S} \cong \bigoplus_\beta \overline{I}_\beta I_\beta$. Since $\overline{R}/\overline{J} \cong R/J$ is semisimple, so is $\overline{S}$, it follows that B is a finite set and R is G-semilocal.

Lemma 1.4 *Let R be any ring, $I = Re_1 \oplus Re_2$, where e_1 and e_2 are idempotents. If $e_2e_1 = 0$, then there is an idempotent e such that $I = Re$.*

Proof Let $\pi \in \mathrm{Hom}_R(R, I)$ with $\pi(r) = r(e_1 + e_2)$. Since $\pi : Re_2 \twoheadrightarrow Re_2$ and $\pi : R(1-e_2) \twoheadrightarrow Re_1$, so that $\pi : R \to I$ is epimorphic. Let $M = \mathrm{Ker}\pi$, then there is an exact sequence $0 \to M \to R \xrightarrow{\pi} I \to 0$. Since $\pi : r(e_1+e_2-e_1e_2) \mapsto r(e_1+e_2) \in I$, thus $r - r(e_1+e_2-e_1e_2) \in M$. So that for any $r \in R$, we have $r = [r - r(e_1+e_2-e_1e_2)] + r(e_1+e_2-e_1e_2)$, it follows that $R = M + I$. We shall now prove that $M \cap I = 0$. If $x \in M \cap I$, then there are $r_i (i = 1, 2)$ such that $x = r_1e_1 + r_2e_2$. It follows that $0 = \pi(x) = r_1e_1 + (r_1e_1 + r_2)e_2$, this implies $r_1 = r_2 = 0$, hence $x = 0$, we have thus proved that $M \cap I = 0$ and $R = M \oplus I$. The lemma follows by [3,Proposition 7.1].

Proposition 1.5 *Let R be a G-semilocal ring and $S = (S \cap J) \oplus (\oplus_{i=1}^n I_i)$ with $I_i^2 = I_i$, then there is an idempotent e such that $Re = \oplus_{i=1}^n I_i$.*

Proof Let $S_1 = \oplus_{i=1}^n I_i$. It is easy to show that there is a maximal left ideal M_i such that $R = I_i \oplus M_i$ for every I_i. If there is an $I_i (i > 1)$ with $I_i \not\subset M_1$, then we have $R = M_1 \oplus I_i$, put $I = I_1 \oplus I_i$, then $R = I + M_1$, it follows from [3, Lemma 9.2] that $I = (I \cap M_1) \oplus I_t (t = 1 \text{ or } i)$. Now let L_i denote $I \cap M_1$ and $L_1 = I_t$, then we get that $I = L_1 \oplus L_i$ with $L_i \subset M_1$, clearly $L_i = I \cap M_1$ is a minimal left ideal. If $L_i^2 = 0$, then $L_i \subset S \cap J$, so that $I_i \subset (S \cap J) \cap S_1$ and this contradicts that $S = (S \cap J) \oplus S_1$, thus $L_i^2 = L_i$. If $j \neq 1, i$, we let $L_j = I_j$, then we can get that $S = \oplus_{i=1}^n L_i$ with $L_1 \not\subset M_1$ and $L_i \subset M_1$(for $i > 1$), where $L_j (j = 1, 2, \cdots, n)$ are minimal left ideals with $L_j^2 = L_j$.

In general, for an index set $\{I_i, I_{i+1}, \cdots, I_n\} (i > 1)$ with $I_j \subset M_{i-1} (1 \leq j \leq n)$. We note that $R = I_i \oplus M_i$, and if there is an $I_j (j > i)$ with $I_j \not\subset M_i$, then $R = M_i \oplus I_j$. Let $I = I_i \oplus I_j$, we get that $R = I + M_i$, it follows that $I = (I \cap M_i) \oplus I_t$ ($t = 1$ or j). Let L_j denote $I \cap M_i$ and $L_i = I_t$ then $I = L_i \oplus L_j$. By the proof as above it follows that $L_j^2 = L_j$ is a minimal left ideal. Also we note that $L_j \subset I = I_i \oplus I_j \subset M_{i-1}$.

Therefore, without loss of generality it may be assumed that there is an indexed set of minimal left ideal $\{L_1, L_2, \cdots, L_n\}$ such that

(a) $S_1 = \oplus_{i=1}^n L_i$ and $L_i^2 = L_i (i = 1, 2, \cdots, n)$; (b) $L_i \not\subset M_i, L_j \subset M_i$ for $j > i$.

Since $R = L_i + M_i$, so that there is an idempotent $e_i \in L_i$ such that $L_i = Re_i$ and $M_i = R(1 - e_i)$, and it is clear that $M_ie_i = 0$, then $e_je_i = 0 (j > i)$. It follows from Lemma 1.4 that there is an idempotent u_{n-1} such that $L_{n-1} \oplus L_n = Ru_{n-1}$. Since $u_{n-1} \in L_{n-1} \oplus L_n \subset M_{n-2}$, then $u_{n-1}e_{n-2} = 0$. Again, by Lemma 1.4, there is an idempotent u_{n-2} such that $Ru_{n-2} = L_{n-2} \oplus Ru_{n-1} = L_{n-2} \oplus L_{n-1} \oplus L_n$. By the same way to get $\oplus_{i=1}^n L_i = Re$ where $e = u_1$ is an idempotent. □

2. Homological Dimensions

Theorem 2.1 *Let R be a G-semilocal ring, $S = \mathrm{Soc}({}_RR)$ and $J = J(R)$, then*

$$\mathrm{l.gd}(R/S) = \mathrm{l.gd}(R/S \cap J) \quad \text{and} \quad \mathrm{r.gd}(R/S) = \mathrm{r.gd}(R/S \cap J).$$

Proof Let $\overline{R} = R/S \cap J$, then by Proposition 1.5 there is an idempotent $e \in R$ such that $\overline{S} = \overline{R}\bar{e}$. Let $\overline{M} = \overline{R}(\bar{1} - \bar{e})$, then $\overline{R} = \overline{S} \oplus \overline{M}$.

We shall now prove that $\overline{M}$ is also a two-sided ideal of $\overline{R}$. It is sufficient to show that $\overline{MS} = 0$, and then $\overline{MR} = \overline{M}(\overline{S} + \overline{M}) \subset \overline{MS} + \overline{M}^2 = \overline{M}^2 \subset \overline{M}$.

Let $\overline{MS} \neq 0$. Since $\overline{S} = \oplus_{i=1}^n I_i$, and $\bar{I}_i^2 = \bar{I}_i$ is a minimal left ideal for every i, then there is an $\bar{I}_i$ such that $\overline{M}\bar{I}_i = \bar{I}_i$. We note that $\overline{SM} = 0$ and $\overline{S}\bar{I}_i = \bar{I}_i$, then

$$\bar{I}_i = \overline{S}\bar{I}_i = \overline{S}(\overline{M}\bar{I}_i) = 0 \cdot I = 0$$

which is a contradiction.

Now by [3, Proposition 7.6] it follows that $\overline{R} = \overline{S} + \overline{M}$ is a ring decomposition of $\overline{R}$. From [1.p.190] we have

$$\mathrm{l.gd}(\overline{R}) = \max\{\mathrm{l.gd}(\overline{S}),\ \mathrm{l.gd}(\overline{M})\} \quad \text{and} \quad \mathrm{r.gd}(\overline{R}) = \max\{\mathrm{r.gd}(\overline{S}),\ \mathrm{r.gd}(\overline{M})\}.$$

Since $\overline{S} = \oplus_{i=1}^n I_i$ is semisimple, so that $\mathrm{l.gd}(\overline{R}) = \mathrm{l.gd}(\overline{M})$. We note that $\overline{M} \cong \overline{R}/\overline{S} \cong R/S$, then $\mathrm{l.gd}(R/S \cap J) = \mathrm{l.gd}(R/S)$. Similarly, we have $\mathrm{r.gd}(R/S \cap J) = \mathrm{r.gd}(R/S)$.

This result implies immediately the following

Corollary 2.2 *Let R be a semilocal ring, then*

$$\mathrm{l.gd}(R/S) = \mathrm{l.gd}(R/S \cap J) \quad \text{and} \quad \mathrm{r.gd}(R/S) = \mathrm{r.gd}(R/S \cap J).$$

Corollary 2.3 *Let R be a G-semilocal ring, if $S \cap J = 0$ (particularly, when R is a semiprimitive ring), then* $\mathrm{l.gd}(R/S) = \mathrm{l.gd}(R)$ *and* $\mathrm{r.gd}(R/S) = \mathrm{r.gd}(R)$.

Theorem 2.4 *Let R be a G-semilocal ring, then* $\mathrm{r.gd}(R) \leq \mathrm{r.gd}(R/S) + l.fd_R(R/S)$.

Proof We know that $J = J(R)$ annihilates all simple left ideals, then $JS = 0$. It follows that $S \cap J$ is a nilpotent ideal of R, then by [6, Theorem 1] we have

$$\mathrm{r.gd}(R) \leq \mathrm{r.gd}(R/S \cap J) + \mathrm{l.fd}_R(R/S \cap J).$$

We note that $\mathrm{r.gd}(R/S \cap J) = \mathrm{r.gd}(R/S)$ by Theorem 2.1. Let $\overline{R} = R/S \cap J$, then by proposition 1.5 there is an idempotent e such that $\overline{S} = \overline{R}\bar{e}$, then

$$\overline{R} = \overline{S} \oplus \overline{M} \quad \text{where} \quad \overline{M} = \overline{R}(\bar{1} - \bar{e}).$$

Since $\overline{S} \cong \oplus_{i=1}^n I_i$, and $I_i^2 = I_i$ is a minimal left ideal for every i, it is easy to get that $\mathrm{l.pd}_R I_i = 0$, then $\mathrm{l.pd}_R\overline{S} = 0$, and so that $\mathrm{l.fd}_R\overline{R} = \mathrm{l.fd}_R\overline{M} = \mathrm{l.fd}(\overline{R}/\overline{S}) = \mathrm{l.fd}_R(R/S)$.

Remark In [6] Thomas has shown that for any ring R

$$\mathrm{l.gd}(R) \leq \mathrm{l.gd}(R/S) + \mathrm{l.pd}_R(R/S).$$

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G- 半局部环及其同调维数

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摘 要

本文中讨论了一类比半局部环更广的环类，即G- 半局部环. 对G- 半局部我们通过模去环的左Socle 及Jacobson 根，研究了环的同调维数，并得到$Gd(R/S) = Gd(R/S \cap J)$，式中的Gd 表示环R 的左整体维数或右整体维数，$S = \mathrm{Soc}({}_RR)$ 以及J 是环R 的Jacobson 根. 当R 还是半本原环时，即得$Gd(R/S) = Gd(R)$.