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A Construction of MATCH (14,3,1)-Designs *

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Abstract A MATCH (n,k,λ)-design is a collection of k-matchings of the complete graph K_n with the property that every pair of independent edges lies in exactly λ members of the collection. In this paper, we shall construct a MATCH (14,3,1)-design, which is an open problem in [1].

Keywords complete graph, 1-factorization, block design, Latin square.

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Alspach and Heinrich [1] introduced the concept of matching designs as a generalization of hyperfactorizations by Jungnickel and Vanstone [3]. A MATCH (n,k,λ)-design is a collection of k-matchings (i.e., k independent edges) of the complete graph K_n (repetitions are allowed) such that every pair of independent edges lies in exactly λ members of the collection. As a matter of fact, a MATCH (n,k,λ)-design is simply a triangular PBIBD on the edge set of K_n with parameters $\lambda_1=0$ and $\lambda_2=1$.

A matching design is called simple if it has no repeated k-matchings, and trivial if it is a multiple of all k-matchings of K_n. It is obvious that every MATCH $(n,k,1)$-design, if exists, is simple.

It is simple to show that the number of k-matchings in a MATCH (n,k,λ)- design is

$$b=\lambda n(n-1)(n-2)(n-3)/4k(k-1), \tag{1}$$

and the number of k-matchings in which a particular edge lies is

$$r=\lambda(n-2)(n-3)/2(k-1). \tag{2}$$

So the divisibility conditions

$$4k(k-1)|\lambda n(n-1)(n-2)(n-3), \tag{3}$$

and

$$2(k-1)|\lambda(n-2)(n-3) \tag{4}$$

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are tow necessary conditions for the existence of a MATCH (n,k,λ)-design. With $k=3$ and $\lambda=1$, the conditions (3) and (4) imply that $n \equiv 2$ or 3 (mod4). For $n=6$, there exists a trivial MATCH $(6,3,1)$-design. For $n=7$, it is easy to prove that no MATCH $(7,3,1)$-design exists. For $n=11$ and 15, both a MATCH $(11,3,1)$-design and a MATCH $(15,3,1)$-design have been constructed in [1]. So 10 and 14 are the smallest two admissible values for which the existence of corresponding MATCH $(n,3,1)$-designs has not been determined yet so far. In the present paper, we shall give a construction of MATCH $(14,3,1)$-design by means of the technique of Latin squares. And as a consequence of theorem 2.8 in [1], we can also get a MATCH $(42,3,1)$-design.

Theorem 1 *There exists* a MATCH $(14,3,1)$-*design.*

Proof Let the vertex set of K_{14} be $\{\infty,1,2,\cdots,13\}$. It is well known that K_{14} has a 1-rotational-factorization GK_{14} (see [4]). Denote by $F_1,F_2,\cdots,F_{13}$ the thirteen 1-factors, where F_i contains edge $(i\ \ \infty)$ for all i in the range $1\le i\le 13$.

Step 1 Take a BIBD (7,3,1) on the edge set of each of the factors, then each block of the BIBD is a 3-matching of K_{14}. There are all together $13\times 7=91$ 3-matchings obtained this way, which contain all possible pairs of independent edges within each of the factors.

Step 2 Taking a BIBD (13,3,1) on the factor set of the 1-factorization gives 26 blocks $\{F_i,F_{i+2},F_{i+8}\}$ and $\{F_i,F_{i+3},F_{i+4}\}$, $i=1,2,\cdots,13$ (subscripts mod 13).

Step 3 Construct thirty five 3-matchings from block $\{F_1,F_3,F_9\}$ by the following procedure.

1) Define a 7×7 partial Latin square $A=(a_{ij})$ as shown by boldface entries in Figure 1, where the rows of the square stand for the edges in F_9 and the columns for the edges in F_3. The entry $a_{ij}=1$ if edge $(1\ \ \infty)$ in F_1 is adjacent to both ith row and jth column at either vertex 1 or ∞, the entry $a_{ij}=k$ $(k=2,3,\cdots,7)$ if edge $(k\ \ 15-k)$ in F_1 is adjacent to both ith row and jth column at either vertex k or $15-k$.

	(3∞)	(2 4)	(1 5)	(6 13)	(7 12)	(8 11)	(9 10)
(2 3)	**3**	**2**					
(1 4)		**4**	**1**				
(5 13)			**5**	**2**			
(6 12)				**6**	**3**		
(7 11)					**7**	**4**	
(8 10)						**7**	**5**
(9∞)	**1**						**6**

Figure 1

2) Complete the partial Latin square in 1) to a Latin square as shown in Figure 2.

	(3∞)	(2 4)	(1 5)	(6 13)	(7 12)	(8 11)	(9 10)
(2 3)	**3**	**2**	4	7	5	6	1
(1 4)	7	**4**	**1**	5	6	2	3
(5 13)	6	7	**5**	**2**	1	3	4
(6 12)	4	5	2	**6**	**3**	1	7
(7 11)	5	6	3	1	**7**	**4**	2
(8 10)	2	1	6	3	4	**7**	**5**
(9∞)	1	3	7	4	2	5	**6**

Figure 2

3) According to the correspondence between the edges of K_{14} and the symbols, rows and columns of entries given in 1), each filled entry corresponds to a 3-matching. Thus we get thirty five 3-matchings as follows:

(1∞,2 4,8 10) (1∞,6 13,7 11) (1∞,7 12,5 13) (1∞,8 11,6 12) (1∞,9 10,2 3)
(2 13,3∞,8 10) (2 13,1 5,6 12) (2 13,7 12,9∞) (2 13,8 11,1 4) (2 13,9 10,7 11)
(3 12,2 4,9∞) (3 12,1 5,7 11) (3 12,6 13,8 10) (3 12,9 10,1 4) (3 12,8 11,5 13)
(4 11,3∞,6 12) (4 11,1 5,2 3) (4 11,6 13,9∞) (4 11,7 12,8 10) (4 11,9 10,5 13)
(5 10,3∞,7 11) (5 10,2 4,6 12) (5 10,6 13,1 4) (5 10,7 12,2 3) (5 10,8 11,9∞)
(6 9,3∞,5 13) (6 9,2 4,7 11) (6 9,1 5,8 10) (6 9,7 12,1 4) (6 9,8 11,2 3)
(7 8,3∞,1 4) (7 8,2 4,5 13) (7 8,1 5,9∞) (7 8,6 13,2 3) (7 8,9 10,6 12)

Step 4 Using the similar way in Step 3, we get thirty five 3- matchings from block $\{F_1, F_4, F_5\}$ as follows:

(1∞,2 6,3 7) (1∞,8 13,11 12) (1∞,10 11,2 8) (1∞,9 12,10 13) (1∞,3 5,4 6)
(2 13,4∞,1 9) (2 13,10 11,4 6) (2 13,9 12,3 7) (2 13,1 7,5∞) (2 13,35,11 12)
(3 12,4∞,10 13) (3 12,2 6,5∞) (3 12,8 13,4 6) (3 12,10 11,1 9) (3 12,1 7,2 8)
(4 11,2 6,1 9) (4 11,8 13,3 7) (4 11,9 12,5∞) (4 11,17,10 13) (4 11,3 5,2 8)
(5 10,4∞,3 7) (5 10,2 6,11 12) (5 10,8 13,1 9) (5 10,9 12,2 8) (5 10,1 7,4 6)
(6 9,4∞,2 8) (6 9,8 13,5∞) (6 9,10 11,3 7) (6 9,17,11 12) (6 9,3 5,10 13)
(7 8,4∞,11 12) (7 8,2 6,10 13) (7 8,10 11,5∞) (7 8,9 12,4 6) (7 8,3 5,1 9)

Step 5 Let σ be the permutation given by $\sigma = (1\ \ 2\ \ 3\ \ \cdots\ \ 13)$. From each 3-matching in Step 3 and Step 4 we get further twelve 3-matchings by acting the permutations σ^i on it, $i = 1, 2, \cdots, 12$.

It is easy to see that any two independent edges of K_{14} must be contained either in one of the 91 3-matchings in Step 1 if they come from the same factor of the 1-factorization, or in one of the 910 3-matchings from Step 3 to Step 5 if they come from two different factors. Hence, the $91 + 910 = 1001$ 3- matchings obtained above form a MATCH $(14, 3, 1)$-design of K_{14}.

To prove the existence of a MATCH $(42, 3, 1)$-design, we need a definition and a lemma by Alspach and Heinrich [1].

Definition *A core of a* MATCH $(n, 3, 1)$*-design is a subset M of 3-matchings in the design such that for every vertex x of K_n, $M_x = \{(ab, cd) : (ab, cd, xy) \in M$ for some vertex y in $K_n\}$ is a partition of all edges of $K_n - x$ into 2-matchings.*

Lemma *If there exists a* MATCH $(n,3,1)$*-design with a core, then there exists a* MATCH $(3n,3,1)$*-design.*

Theorem 2 *There exists a* MATCH $(42,3,1)$*-design.*

Proof Notice that the ninty one 3-matchings in Step 1 in constructing of the MATCH $(14,3,1)$-design form a core of the design. Hence, by Lemma, there exists a MATCH $(42,3,1)$-design.

Remark The technique here used for constructing a MATCH $(14,3,1)$- design can be generalized to K_n, where $n \equiv 2 \pmod{12}$. It is obvious that such a K_n has a 1-rotational-factorization, and for such a K_n, both a BIBD $(n-1,3,1)$ and a BIBD $(n/2,3,1)$ exist. All that need to be shown is that the partial Latin square defined in Step 3 can be completed. In [2], Daykin and Häggkvist conjectured that a partial Latin square of order n in which every row, every column and every symbol is used at most $n/4$ times can be completed to a Latin square. If this conjecture is true, we shall be able to construct a series of MATCH $(n,3,1)$-designs and, as a consequence of Lemma, a series of MATCH $(3n,3,1)$-designs with $n \equiv 2 \pmod{12}$.

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References

[1] B. Alspach and K. Heinrich, *Matching design*, Aust. J. of Combinatorics, **2**(1990), 39—55.

[2] D.E. Daykin and R. Häggkvist, *Completion of sparse partial latin square*, Graph Theory and Combinatorics, Proc. Cambridge Conference in Honour of Paul Erdös, 1983.

[3] D. Jungnickel and S.A. Vanstone, *Hyperfactorizations of graphs and 5-designs,* J. Kuwait Univ. (Sci.), **14**(1987), 213—224.

[4] E. Mendelsohn and A. Rosa, *One-factorizations of the complete graph—a survey*, J. Graph Theory, **9**(1985), 43—65.

MATCH (14,3,1)- 设计的一个构造法

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摘 要

一个MATCH (n,k,λ)- 设计就是完全图 K_n 的一个 k- 匹配集合，使得 K_n 的每一对独立边恰好出现在 λ 个 k- 匹配中．本文构造了一个MATCH (14,3,1)- 设计，解决了文献[1] 中一个尚未解决的问题，同时还得到一个MATCH (42,3,1)- 设计．

关键词 完全图，1- 因子分解，区组设计，拉丁方．