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The Harmonious Chromatic Number of a Complete 4-Ary Tree *

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Abstract In this paper a quite accurate estimate of the harmonious chromatic number of a complete 4-ary tree was given.

Keywords graph, harmonious chromatic number, complete 4-ary tree, adjacent, color.

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The harmonious chromatic number of a graph G, denoted by $h(G)$, is the least number of colors which can be assigned to the vertices of G such that each vertex has exactly one color, adjacent vertices have different colors and any two edges have different color pairs. Here a color pair for edge e is a set of colors on the vertices of e. The harmonious colorings have been studied in papers [1-8].

Finding the harmonious chromatic number of a graph is apparently quite difficult. However, the harmonious chromatic number of complete binary and trinary tree has been estimated quite exactly in [6] and [8]. Here we will discuss the harmonious chromatic number for a complete 4-ary tree with n levels which will be denoted by T_n^4.

Since the edges must have different color pairs it is apparent that if $h(G) = h$, then the binomial coefficient $C(h,2) \geq |E(G)|$. In this paper we let $k = k(G)$ denote the smallest integer such that $C(k,2) \geq |E(G)|$. So, we have $h(G) \geq k(G)$. It is easy to verify that $k(T_n^4) > \sqrt{\frac{2}{3}}4^{\frac{n}{2}}$.

First, we design a partial harmonious coloring U of the upper levels of T_{2r}^4 with 4^r colors. A partial harmonious coloring of a graph G is a harmonious coloring of any induced subgraph of G such that for any uncolored vertex u and any color c, at most one element of $N(u)$ has the color c. Here $N(u)$ is the set of vertices of G adjacent to u.

Coloring U Denote the color set $A_k^{i,j} = \{(j-1)4^{i+1}+4^i(k-1)+l, 1 \geq l \geq 4^i\}, k = 1,2,3,4$. First, we color the upper three level of T_6^4 with 4^3 colors as shown in Fig.1. The number on the left indicates the times that each color appears in this level. The sign $\begin{array}{c}M\\|\\N\end{array}$, where

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M and N are two color sets, indicates that each vetex with color from N adjacents to four vertices with color from M and each color of N is due to adjacent to each color of M once and only once. So by Fig.1. we can see that, for example, each number color occurring 4 times on the level five and 16 endvertices adjacent to each color of $A_k^{2,1} = \bigcup_{i=1}^{4} A_i^{1,k} (k = 1, 2, 3)$ are colored with $A_{k+1}^{2,1}$. We assign the colors to the level four such that each color set $A_k^{1,j}$ is adjacent to the same vertex on the level three and four vertices on the level three adjacent to the same vertex on the level two respectively are adjacent to $A_k^{1,j}, k = 1, 2, 3, 4$, for a same j. Apparently, the partial coloring of upper three levels is harmonious and possesses the property 1: for all i, j the color set pairs $A_1^{i,j}$ and $A_3^{i,j}, A_2^{i,j}$ and $A_4^{i,j}$ respectively are not adjacent to each other.

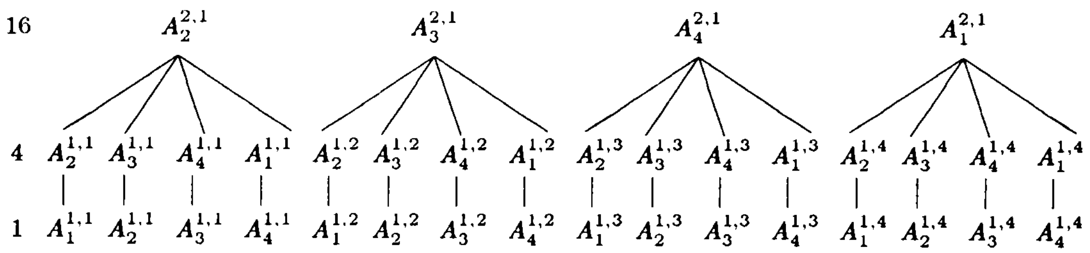

Fig.1 The coloring of the upper three levels of T_6^4

For $r \geq 4$, we will color T_{2r}^4 inductively. Suppose we have a partial coloring of the upper r levels of T_{2r}^4, which possesses the property 1, we now color T_{2r+2}^4. Let $H_k^r, k = 1, 2, 3, 4$ be copies of T_{2r}^4. Color H_1^r as T_{2r}^4 was originally colored. If vertex v of H_1^r is colored with number n, color the corresponding vertex of $H_k^r, k = 2, 3, 4$ with color $n + (k-1)4^r$. Now form T_{2r+2}^4 from $H_k^r, k = 1, 2, 3, 4$, a new vertex which is adjacent to the four roots of $\bigcup_{k=1}^{4} H_k^r$, and 4^{2r+1} new endvertices such that 4 are adjacent to each endvertex of $\bigcup_{k=1}^{4} H_k^r$.

We now color the endvertices of T_{2r+2}^4. Note that the endvertices of H_k^r are colored with $A_k^{r,1}, k = 1, 2, 3, 4$ and each color is used exactly 4^{r-1} times. So, we can assign colors $A_k^{r,1}$ to the 4^r endvertices adjacent to each color in the top level of $H_{k-1}^r, k = 2, 3, 4$ and assign colors $A_1^{r,1}$ to the 4^r endvertices adjacent to each color in the top level of H_4^r.

Notice that $\bigcup_{k=1}^{4} A_k^{i,j} = A_{k'}^{i+1,j'}$, where $k' = (j-1) \pmod 4 + 1, j' = \frac{j-k'}{4} + 1$, the coloring of the upper three levels of T_{2r}^4 is shown in Fig.2.

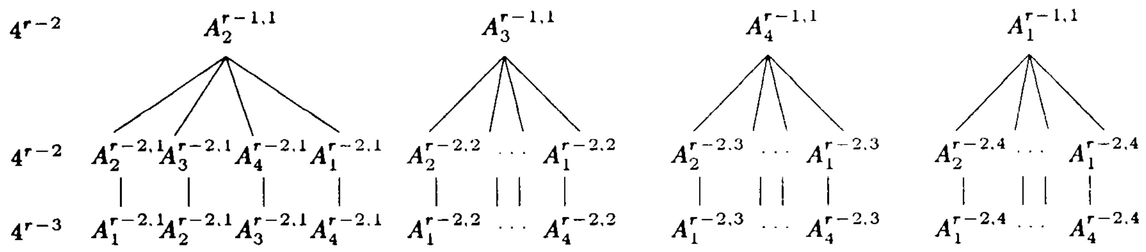

Fig.2 The coloring of the upper three levels of T_{2r}^4

Apparently, the Coloring U is harmonious and possesses the property 1 and property 2: each vertex on the level r is adjacent to a color set $A_k^{1,j}$ on the level $r+1$ and four of them adjacent to the same vertex on the level $r-1$ respectively are adjacent to $A_k^{1,j}, k=1,2,3,4$ for a same j.

Theorem 1 *For* $r \geq 3, \sqrt{\frac{2}{3}}4^r < h(T_{2r}^4) \leq \frac{7}{8}4^r + 5$.

Proof We prefer to color T_{2r+2}^4. Take four copies $H_k^r, k=1,2,3,4$, of T_{2r}^4. First color them as Coloring U. Then use the colors $A_1^{r-1,1} \cup A_3^{r-1,1}$ to substitute for colors $\bigcup_{j=3\cdot 4^{r-2}+1}^{4^{r-1}} (A_1^{1,j} \cup A_3^{1,j})$ to recolor H_4^r. For convenience we write $D_k^{i,j} = A_k^{i,j+3\cdot 4^{r-i-1}}$ to denote the colors originally used in H_4^r. We will make a special arrangement inductively to ensure the color pairs used in the edges of H_1^r will not appear in the edgds of H_4^r. Substitute $A_k^{1,1}$ for $D_k^{1,1}$; $A_k^{1,3}$ for $D_k^{1,2}, k=1,3$ and $A_k^{1,1}$ for $D_{k-1}^{1,3}$; $A_k^{1,3}$ for $D_{k-1}^{1,4}, k=2,4$. In terms of the Property 1 we can see the color pairs arising therefrom have not been used in H_1^r. Then use $A_1^{2,3} \cup A_3^{2,3}, A_2^{2,1} \cup A_4^{2,1}$ and $A_2^{2,3} \cup A_4^{2,3}$ respectively to recolor the corresponding vertices originally colored by $D_2^{3,1}, D_3^{3,1}$ and $D_4^{3,1}$ in the same way. Suppose we have already recolored the related vertices originally colored by $D_1^{i,1}$ with $A_1^{i-1,1} \cup A_3^{i-1,1}, 1 < i < r$, appropriately such that none of the color pairs used in H_1^r appear in H_4^r, Then use $A_1^{i-1,3} \cup A_3^{i-1,3}, A_2^{i-1,1} \cup A_4^{i-1,1}$ and $A_2^{i-1,3} \cup A_4^{i-1,3}$ respectively to recolor the correspounding vertices originally colored by $D_2^{i,1}, D_3^{i,1}$ and $D_4^{i,1}$ in the way that the related vertices originally colored by $D_1^{i,1}$ was recolored. In terms of the Property 1 we can see the color pairs arising therefrom have not been used in H_1^r. By mathematical induction, we get the required recolor of H_4^r.

Now form T_{2r+2}^4 from $H_k^r, k=1,2,3,4$, a new vertex which is adjacent to the four roots of $\bigcup_{k=1}^4 H_k^r$, and 4^{2r+1} new endvertices such that 4 are adjacent to each endvertex of $\bigcup_{k=1}^4 H_k^r$. We can assign colors $A_k^{r,1}$ to the 4^r endvertices adjacent to each color in the top level of $H_{k-1}^r, k=2,3$. Note that the endvertices of H_4^r are colored with $A_1^{r-1,1} \cup$

$A_3^{r-1,1} \cup \bigcup_{j=1}^{4^{r-2}} (D_2^{1,j} \cup D_4^{1,j})$ and each color is used exactly 4^{r-1} times, we assign colors $A_2^{r,1}$ and $A_3^{r,1}$ respectively to the endvertices adjacent to each color of $\bigcup_{j=1}^{4^{r-2}} (D_2^{1,j} \cup D_4^{1,j})$ and of $A_1^{r-1,1} \cup A_3^{r-1,1}$ in the top level of H_4^r. Finally, assign colors $A_2^{r-1,1} \cup A_4^{r-1,1} \cup \bigcup_{j=1}^{4^{r-2}} (D_2^{1,j} \cup D_4^{1,j})$ to the endvertices adjacent to each color in the top level of H_3^r.

We color the level $r+1$ with five special colors $S_k, 1 \le k \le 5$. Respectively assign S_k to the vertices which are adjacent to $A_k^{i,j}, k = 1,2,3$, in $\bigcup_{k=1}^{3} H_k^r$. Assign S_4 to the vertices which are adjacent to $A_4^{1,j}$ in $H_1^r \cup H_3^r$ and S_5 to the vertices which are adjacent to $A_4^{1,j}$ in H_2^r. Because each color set $A_k^{1,j}$ is used at most two times in level $r+2$, so only after a proper rearrangement for the order of $S_k, k = 1,2,3,5$, we can assign S_1, S_2, S_3, S_5 to the four vertices which are adjacent to the same vertex on the level r in H_4^r. The colored part is a partial harmonious coloring in virtue of property 2.

Now we color the level below the level $r+1$ in $H_1^r \cup H_3^r$ with the colors $B = \bigcup_{j=4^{r-2}+1}^{2\cdot 4^{r-2}} A_4^{1,j}$, which are only adjacent to S_5 among the special colors in the previous coloring. Note that the number of the vertices on the level r in H_k^r is just equal to the number of the colors of $B \cap A_k^{r-1,2}$, we assign the colors $B \cap A_3^{r-1,2}$ and $B \cap A_4^{r-1,2}$ respectively to the level r in H_1^r and H_3^r. Then assign the colors $B \cap A_3^{r-2,5}$ and $B \cap A_3^{r-2,6}$ respectively to the level $r-1$ in H_1^r and H_3^r. Generally, we assign the colors $B \cap A_3^{r-i,4^{i-1}+1}$ and $B \cap A_3^{r-i,4^{i-1}+4^{i-2}+1}$ respectively to the level $r-i+1$ in H_1^r and $H_3^r, 2 \le i \le r-2$. Finally, we use one of the colors $A_4^{1,4^{r-2}+1}$ and $A_4^{1,4^{r-2}+4^{r-3}+1}$ respectively to color the root of H_1^r and H_3^r.

Similarly, we color the level below the level $r+1$ in $H_2^r \cup H_4^r$ in the same way with the colors $B = \bigcup_{j=2\cdot 4^{r-2}+1}^{3\cdot 4^{r-2}} A_4^{1,j}$, which are only adjacent to S_4 among the special colors. Finally, we use S_1 to color the root of T_{2r+2}^4. Using the property 1, it is easy to check that the coloring of T_{2r+2}^4 is harmonious and the theorem is proved.

Theorem 2 *For* $r \ge 3, 2\sqrt{\frac{2}{3}}4^r < h(T_{2r+1}^4) \le \frac{27}{16}4^r + 4$.

Proof First color the level above the level r in T_{2r}^4 as coloring U. Note that the 4^{2r-1} endvertices of T_{2r}^4 are colored with 4^r colors, each occurring 4^{r-1} times. Join four new vertices to each endvertex to form T_{2r+1}^4. The coloring of T_{2r+1}^4 is divided into two parts.

A. The coloring of the upper level

We will use $\frac{11}{16} \cdot 4^r$ new colors. Divide them into 11 groups equally and denote them by $E^i = \{4^r + (i-1)4^{r-2} + j, 1 \le j \le 4^{r-2}\}, 1 \le i \le 11$. For convenience we write $E^i = E^j$ for $i = j \pmod{11}$. Now for each $k, 1 \le k \le 4$, use E^k to recolor the endvertices of T_{2r}^4 which have the colors $A_k^{r-2,2}$ and are adjacent to the colors $A_k^{r-2,1}$. Assign the colors

$A_k^{r-2,2}$ and $E^i, i = k+1, k+2, k+3$, to the new endvertices adjacent to each color of E^k. Then for each $k, k = 1, 2$, assign the colors $A_k^{r-2,1}, A_1^{r-2,4}, A_2^{r-2,4}$ and $E^i, i \neq k, k+4$, to the new endvertices adjacent to each color of $A_k^{r-2,2}$ and for each $k, k = 3, 4$, assign the colors $A_k^{r-2,1}, A_3^{r-2,4}, A_4^{r-2,4}$ and $E^i, i \neq k, k+4$, to the new endvertices adjacent to each color of $A_k^{r-2,2}$. Using the Property 1 we can see that the coloring of the previous parts is harmonious.

Because the colors E^{k+4} is not adjacent to the colors $A_k^{r-2,2}$ for each $k, 1 \leq k \leq 4$, in the previous coloring, so we can use E^{k+4} to recolor the endvertices of T_{2r}^4 which have the colors $A_k^{r-2,3}$ and are adjacent to the colors $A_k^{r-2,2}$. Then assign the colors $A_k^{r-2,3}$ and $E^i, i = k+5, k+6, k+7$, to the new endvertices adjacent to each color of E^{k+4}. Meanwhile for each $k, k = 1, 2$, assign the colors $A_k^{r-2,2}, A_1^{r-2,1}, A_2^{r-2,1}$ and $E^i, i \neq k+4, k+8$, to the new endvertices adjacent to each color of $A_k^{r-2,3}$ and for each $k, k = 3, 4$, assign the colors $A_3^{r-2,2}, A_3^{r-2,1}, A_4^{r-2,1}$ and $E^i, i \neq k+4, k+8$, to the new endvertices adjacent to each color of $A_k^{r-2,3}$.

Similary, we use E^{k+8} to recolor the endvertices of T_{2r}^4 which have the colors $A_k^{r-2,4}$ and are adjacent to the colors $A_k^{r-2,3}$. Then for each $k, k = 1, 2, 3$, assign the colors $A_k^{r-2,4}$ and $E^i, i = k+9, k+10, k$, to the new endvertices adjacent to each color of E^{k+8} and assign the colors $A_4^{r-2,4}$ and $E^i, i = 5, 6, 7$, to the new endvertices adjacent to each color of $E^{12} = E^1$. Meanwhile for each $k, k = 1, 2$, assign the colors $A_k^{r-2,3}, A_3^{r-2,2}, A_4^{r-2,2}$ and $E^i, i \neq k+8, k+1$, to the new endvertices adjacent to each color of $A_k^{r-2,4}$ and for each $k, k = 3, 4$, assign the colors $A_k^{r-2,3}, A_1^{r-2,2}, A_2^{r-2,2}$ and $E^i, i \neq k+8, k+1$ to the new endvertices adjacent to each color of $A_k^{r-2,4}$.

Finally use E^{k+1} to recolor the endvertices of T_{2r}^4 which have the colors $A_k^{r-2,1}$ and are adjacent to the colors $A_k^{r-2,4}$. Then assign the colors $A_k^{r-2,1}$ and $E^i, i = k+5, k+6, k+7$, to the new endvertices adjacent to each color of E^{k+1}. Meanwhile for each $k, k = 1, 2$, assign the colors $A_k^{r-2,4}, A_3^{r-2,3}, A_4^{r-2,3}$ and $E^i, i \neq k+1, k$, to the new endvertices adjacent to each color of $A_k^{r-2,1}$ and for each $k, k = 3, 4$, assign the colors $A_k^{r-2,4}, A_1^{r-2,3}, A_2^{r-2,3}$ and $E^i, i \neq k, k+1$, to the new endvertices adjacent to each color of $A_k^{r-2,1}$. It can be checked that the coloring of the colored part of T_{2r+1}^4 is harmonious.

B. The coloring of the lower level

We will use the fact that two color groups E^2 and E^9 are not adjacent to each other in the coloring above. First, for each vertex on level $r-1$ color its 4 level r neighbors with the colors S_1, S_2, S_3 and S_4. Then color the vertices on level $r-1$ with the colors of E^2 and assign the colors of E^9 to the $4^{r-3} + 4^{r-4} + \cdots + 1 = \frac{1}{3}(4^{r-2} - 1)$ remaining vertices. Now all the vertices of T_{2r+1}^4 are colored harmoniously. The theorem is proved.

Remark Using previous method we can get $14 \leq h(T_4^4) \leq 16$ and $27 \leq h(T_5^4) \leq 30$.

References

[1] D.G.Beane, N.L.Biggs and B.J.Wilson, *The growth rate of the harmonious chromatic number*, J.Graph Theory 13(1989), 291-298.

[2] O.Frank, F.Harary and M.Plantholt, *The line- distinguishing chromatic number of a graph* , Ars Combin, 14(1982), 241-252.
[3] J.Hopcroft and M.S.Krishnamoorthy, *On the harmonios colorings of graphs*, SIAM J. AlG. Disc. Math., 4(1983), 306-311.
[4] Sin-Min Lee and John Mitchem, *An upper bound for the harmonious chromatic number of a graph*, J.Graph Theory, 11(1987), 565-567.
[5] Lu Zhikang, *On an upper bound for the harmonious chromatic number of a graph*, J.Graph Theory, 15(1991), 345-347.
[6] Lu Zhikang, *The harmonious chromatic number of a complete binary and trinary tree*, Discrete Math. 118(1993), 165-172.
[7] Z.Miller and D.Pritikin, *The harmonious coloring number of a graph*, Conqr. Number., 63(1988), 213-228.
[8] J.Mitchem, *On the harmonious chromatic number of a graph*, Discrete Math., 74(1989), 151-157.

四元完全树的调和着色数

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摘 要

本文给出了四元完全树的调和着色数的上界估计.