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Concerning a Kind of Integrals of Complex-Valued Functions of Large Numbers *

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Abstract Here presented are two limit theorems for a kind of integrals involving complex-valued functions of large numbers. The form of integrals may be regarded as a natural generalization of those integrals occured in the probability limit theorems of Chung and Erdös. Our main result is Theorem 2 whose proof rests upon some known results of [1] and makes an extentive use of Bonnet's second mean value theorem and related analytic techniques.

Keywords condition (0), (UB)-property, Bonnet's theorem.

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1. Introduction

Let X be a random variable which assumes only integer values

$$P(X=k)=p_k,\quad p_k>0,\quad \sum_{-\infty}^{\infty} p_k=1.$$

It is assumed that the sequence $\{p_k\}(-\infty<k<\infty)$ satisfies the usual conditions for the first moment and the mean

$$\sum_{-\infty}^{\infty}|k|p_k<\infty,\quad \sum_{-\infty}^{\infty}kp_k=0. \tag{0}$$

The characteristic function of the distribution function of X is given by

$$f(x)=\sum_{-\infty}^{\infty}p_k e^{ikx},\quad (i=\sqrt{-1}) \tag{1}$$

Consider the sum of n random variables $S_n=\sum_1^n X_k$, where the X_k's are mutually independent, each having the same distribution as X. Then for every integer a,

$$P(S_n=a)=\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x))^n e^{-iax}dx. \tag{2}$$

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One of the two main results of [1] is the following theorem (cf. Theorem 3.1, loc. cit.)

$$\lim_{n\to\infty} P(S_n = a)/P(S_n = a') = 1, \tag{3}$$

where a and a' are any two given integers.

As may be seen, $f'(0) = 0, \max_x |f(x)| = f(0) = 1$ and $|f(x)|^n$ is a peak function rapidly decreasing away from $x = 0$ when n becomes large. Some precise desription of the behavior of $|f(x)|^n$ will be given by some lemmas (cf. §3). Notice that the condition (0) guarantees merely the existence of the first derivative $f'(x)$, but not the existence of $f''(x)$ at $x = 0$. Thus, neither the classical Laplace asymptotic method nor the method of stationary phase (see, e.g., those methods expounded in some detail in Olver's book [4] or in our book [3]) can be used to estimate the integral on the RHS of (2) for $n \to \infty$. In fact, the limit relation (3) under condition (0) was proved using a kind of probabilistic argument (cf. loc. cit.). This may be the reason why one of the authors of [1] has once again posed the problem [2] of seeking a purely analytic proof of (3). As we have learned, this problem is still open till now.

Let us denote

$$J_n(f,g) = \int_{-\pi}^{\pi} (f(x))^n g(x) dx, \tag{4}$$

where $g(x)$ is a continuous function. Suppose that $h(x)$ is also a continuous function defined on $[-\pi, \pi]$ with $h(0) \neq 0$. From some viewpoint of asymptotics, Roderick Wong (Professor of Mathematics at the University of Manitoba, Canada) has conceived and remarked that the limit relation

$$\lim_{n\to\infty} \frac{J_n(f,g)}{J_n(f,h)} = \frac{g(0)}{h(0)} \tag{5}$$

may be true under certain general conditions about f, g and h. This is actually one of the motivations for the present investigation. However, it seems very difficult or even impossible to extend Chung-Erdös'result (3) to the general form (5) under the sole condition (0), even using (3) itself as one of our lemmas. What we can show is that for any two functions $g(x)$ and $h(x)$, analytic in a neighborhood of $x = 0$ with $g'(0) = h'(0) = 0$ and $h(0) \neq 0$, the limit relation (5) is true if $f(x)$ satisfies the condition (0) plus an additional supposition so-called the"uniform boundedness condition" for $\int (f(x))^n dx$. Precise statements will be given in the next section.

2. Statement of Theorems

Hereafter we will use A, A_1 and A_2 to denote some unspecified positive constants, not depending on any variable parameters related.

If the characteristic function $f(x)$ is symmetrical, viz. $p_k = p_{-k}$, so that it becomes a real-valued function, then everything becomes greatly simplified, and an easier result may be stated as follows

Theorem 1 *Let $f(x)$ be symmetrical, and let $g(x)$ be a complex-valued continuous function defined on $[-\pi, \pi]$ with $g(0) \neq 0$. Then we have*

$$\lim_{n\to\infty} n|J_n(f,g)| = \infty \tag{6}$$

$$\lim_{n\to\infty} \sqrt{n}|J_n(f,g)| < A. \tag{7}$$

Clearly the results given by Theorem 1 and Theorem 2.1 of [1] just correspond to the case $g(x) = e^{-iax}$. The proof of (6) and (7) will be omittedd here, since it is very similar to that of the original results in [1]. So we will confine ourselves to proving a much more difficult result Theorem 2, which will be stated afterwards.

Hence-forth we consider the general case where $f(x)$ may be not symmetrical, but subject to the condition (0).

Definition *Let $\Delta_n(t)$ denote the integral average of the real part $\Re(f(x))^n$ on $(-t,t) \subset (-\pi,\pi)$, namely*

(i)
$$\Delta_n(t) := \int_{-t}^{t} \Re(f(x))^n dx.$$

Then $\Delta_n(t)$ is said to have the"uniform boundedness property" ((UB)- property), if for every small $t > 0$, the following inequality holds uniformly for all large n and every t' with $0 < t' < t$:

(ii)
$$0 < \Delta_n(t')/\Delta_n(t) < A.$$

As we shall see later (cf. §3), $\Delta_n(t)$ is positive for all sufficiently large n, and it is asymptotically equivalent to $\int_{-\pi}^{\pi}(f(x))^n dx (n \to \infty)$, since $\Re(f(x))^n$ is also a peak function. Thus the geometrical meaning of (i) and (ii) appears quite apparent.

Theorem 2 *Let $f(x)$ satisfy the condition (0) and let $\Delta_n(t)$ have the (UB)-property. If $g(x)$ is a real-valued function, continuous on $[-\pi,\pi]$ and analytic in a neighborhood of $x = 0$ with $x = 0$ as its stationary point, viz. $g'(0) = 0$, then*

$$\lim_{n\to\infty} \frac{J_n(f,g)}{J_n(f,1)} = g(0). \tag{8}$$

This theorem implies the following tow consequences:

Corollary 1 *Let $g_1(x)$ and $g_2(x)$ be real continuous functions defined on $[-\pi,\pi]$ and satisfy the same conditions as that for $g(x)$ of Theorem 2. Then the limit relation (8) holds for the complex- valued function $g(x) = g_1(x) + ig_2(x)$.*

Corollary 2 *Let $g(x) = g_1(x) + ig_2(x)$ and $h(x) = h_1(x) + ih_2(x)$ be two complex-valued functions with g_1, g_2, h_1 and h_2 satisfying the same conditions as that for $g(x)$ of Theorem 2. Then there holds the limit relation (5) with $h(0) \neq 0$.*

It is easily seen that (5) implies the form of (3) by taking $g(x) = e^{-iax} + iax$ and $h(x) = e^{-ia'x} + ia'x$ and showing $J_n(f,iax)/J_n(f,1) \to 0 (n \to \infty)$.

3. Lemmas with Some Consequences

Lemma 1 *For $f(x)$ defined by (1) we have*

$$|f(x)|^2 = 1 - \sum_{k\neq j} p_k p_j \sin^2(\frac{(k-j)x}{2}),$$

so that $|f(x)|^2$ is an even function.

Lemma 2 *There exist bounding functions for* $|f(x)|^2$*, namely*

$$(1 - A_1 x) \le |f(x)|^2 < (1 - A_2 x^2), \quad (-\pi \le x \le \pi).$$

Lemma 3 *Under the conditin (0) we have for every small number* $\epsilon > 0$

$$P(S_n = a) \ge (1 - \epsilon)^n$$

if $n \ge N(\epsilon, a), a$ *being an integer.*

Both Lemma 1 and Lemma 2 are easily proved using elementary computations (cf. Proof of Theorem 1 in [1]). Lemma 3 is precisely Theorem 2.2 of [1], of which the first proof was given by W.H.J. Fuchs (see, e.g., page 6 of [1]).

Corollary 3 *For every small* $\epsilon > 0$*, we have*

$$J_n(f, 1) \sim \int_{-\epsilon}^{\epsilon} (f(x))^n dx > 0 \quad (n \to \infty) \tag{9}$$

Proof For any small $\epsilon > 0$, by Lemma 2 we find

$$\begin{aligned} \int_{[-\pi,\pi]\setminus(-\epsilon,\epsilon)} (f(x))^n dx &\le \int_{[-\pi,\pi]\setminus(-\epsilon,\epsilon)} (1 - Ax^2)^{n/2} dx < (2\pi)(1 - A\epsilon^2)^{n/2} \\ &< (2\pi)(1 - \frac{1}{2}A\epsilon^2)^n = o((1 - (\frac{1}{2}A\epsilon^2)^2)^n), \quad (n \to \infty). \end{aligned}$$

Thus one may choose $\epsilon_1 = (\frac{1}{2}A\epsilon^2)^2$, so that by Lemma 3, we have for $n \ge N(\epsilon_1)$ with $a = 0$

$$\int_{-\pi}^{\pi} (f(x))^n dx = \int_{-\epsilon}^{\epsilon} + \int_{[-\pi,\pi]\setminus(-\epsilon,\epsilon)} (f(x))^n dx \ge (2\pi)(1 - A\epsilon_1^2)^n.$$

Hence it follows that

$$\int_{-\pi}^{\pi} (f(x))^n dx \sim \int_{-\pi}^{\pi} (f(X))^n dx \quad (n \to \infty). \quad \square$$

Let us denote

$$f(x) = |f(x)| e^{i\theta(x)}, \quad \theta(x) = \arg f(x) = \operatorname{arctg}(\frac{p(x)}{q(x)}),$$

where $p(x) = \sum_{-\infty}^{\infty} p_k \sin kx$ and $q(x) = \sum_{-\infty}^{\infty} p_k \cos kx$. Obviously $\theta(x)$ is an odd function, $\theta(-x) = -\theta(x)$, and so is the function $\sin n\theta(x)$. Thus one may write, for every small $\epsilon > 0$,

$$\begin{aligned} \int_{-\epsilon}^{\epsilon} (f(x))^n dx &= \int_{-\epsilon}^{\epsilon} |f(x)|^n (\cos n\theta(x) + i \sin n\theta(x)) dx \\ &= \int_{-\epsilon}^{\epsilon} |f(x)|^n \cos n\theta(x) dx \equiv \int_{-\epsilon}^{\epsilon} \Re(f(x))^n dx. \end{aligned} \tag{10}$$

From (9) and (10) it is clear that $\int_{-\epsilon}^{\epsilon} \Re(f(x))^n dx$ is always a positive quantity representing the principal part of $\int_{-\pi}^{\pi}(f(x))^n dx$ for n large. This is consistent with (ii) for the (UB)- property.

Lemma 4 *For every small $\epsilon > 0$ we have for $n \to \infty$*

$$\int_{-\epsilon}^{\epsilon} |f|^n \cos n\theta(x) \cos ax dx \sim \int_{-\epsilon}^{\epsilon} |f|^n \cos n\theta(x) dx \tag{11}$$

$$\int_{-\epsilon}^{\epsilon} |f|^n \sin n\theta(x) \sin ax dx = o\left(\int_{-\epsilon}^{\epsilon} |f|^n \cos n\theta(x) dx\right), \tag{12}$$

where a is any integer, not zero.

Proof From (3) and Corollary 3 it follows that

$$\int_{-\pi}^{\pi} (f(x))^n e^{iax} dx \sim \int_{-\pi}^{\pi} (f(x))^n dx \sim \int_{-\epsilon}^{\epsilon} (f(x))^n dx, \quad (n \to \infty).$$

Using Lemmas 2 and 3, one easily gets

$$\int_{-\pi}^{\pi} (f(x))^n e^{iax} dx \sim \int_{-\epsilon}^{\epsilon} (f(x))^n e^{iax} dx.$$

Thus it follows that (via (10))

$$\int_{-\epsilon}^{\epsilon} |f(x)|^n (\cos n\theta(x) + i \sin n\theta(x))(\cos ax + i \sin ax) dx \sim \int_{-\epsilon}^{\epsilon} |f(x)|^n \cos n\theta(x) dx. \tag{13}$$

Since $\cos n\theta(x) \cos ax$ and $\sin n\theta(x) \sin ax$ are even functions, while $\cos n\theta(x) \sin ax$ and $\sin n\theta(x) \cos ax$ are odd functions, it is clear that (13) may be rewritten in the form

$$\int_{-\epsilon}^{\epsilon} |f|^n \cos n\theta(x) \cos ax dx - \int_{-\epsilon}^{\epsilon} |f|^n \sin n\theta(x) \sin ax dx \sim \int_{-\epsilon}^{\epsilon} |f|^n \cos n\theta(x) dx. \tag{14}$$

Changing a into $-a$, one gets

$$\int_{-\epsilon}^{\epsilon} |f|^n \cos n\theta(x) \cos ax dx + \int_{-\epsilon}^{\epsilon} |f|^n \sin n\theta(x) \sin ax dx \sim \int_{-\epsilon}^{\epsilon} |f|^n \cos n\theta(x) dx. \tag{14'}$$

Thus, comparing (14)′ with (14), one obtains (11) and (12). □

We will make use of the Weierstrass form of Bonnet's mean value theorem: Let $F(x)$ be continuous on $[a,b]$ and let $\varphi(x)$ be monotonic in (a,b). Then there is a $\xi, a \le \xi \le b$, such that

$$\int_a^b F(x)\varphi(x) dx = \varphi(a) \int_a^{\xi} F(x) dx + \varphi(b) \int_{\xi}^{b} F(x) dx,$$

where the RHS of the equation will consist of one term if $\varphi(a) = 0$.

4. Proof of Theorem 2

Our proof consists of two usual steps. Firstly, for every given small $\epsilon > 0$, we shall establish certain order relations (involving ϵ) for $n \to \infty$. The second step is to get the desired result by making $\epsilon \to 0_+$.

As may be observed, in order to prove Theorem 2, it suffices to show that, for every given $\epsilon > 0$ there hold the following relations, when $n \to \infty$,

$$\int_{-\epsilon}^{\epsilon} |f|^n \cos n\theta(x) g(x) dx = (g(0) + O(\epsilon)) \int_{-\epsilon}^{\epsilon} |f|^n \cos n\theta(x) dx \tag{15}$$

$$\int_{-\epsilon}^{\epsilon} |f|^n \sin n\theta(x) g(x) dx = O(\epsilon) \int_{-\epsilon}^{\epsilon} |f|^n \cos n\theta(x) dx \tag{16}$$

where the factors involved in the order terms $O(\epsilon)$ are bounded for all large n, i.e., $|O(\epsilon)| < A\epsilon$ for large n. Indeed, a linear combination of (15) and (16) gives

$$\begin{aligned} \int_{-\epsilon}^{\epsilon} |f|^n e^{in\theta(x)} g(x) dx &= [g(0) + O(\epsilon)] \int_{-\epsilon}^{\epsilon} |f|^n \cos n\theta(x) dx \\ &= [g(0) + O(\epsilon)] \int_{-\epsilon}^{\epsilon} |f|^n e^{in\theta(x)} dx \end{aligned}$$

This implies the following with the aid of Lemma 2 and Corollary 3

$$|g(0) - O^+(\epsilon)| \le \overline{\lim_{n\to\infty}} \frac{J_n(f,g)}{J_n(f,1)} \le [g(0) + O^+(\epsilon)],$$

where $O^+(\epsilon) > 0$. Finally (8) will be obtained by letting $\epsilon \to 0_+$.

In what follows we will verify (15) and (16). Let us denote

$$\varphi(x) = g(x) - g(0), \tag{17}$$

where $\varphi(x)$ is not a constant, with $\varphi(0) = 0$ and $\varphi'(0) = 0$. Since $g(x)$ is analytic in a neighborhood of $x = 0$, it is clear that $\varphi'(x)$ should have a definite sign within a small interval $(0, \epsilon)$. The same is true for $\psi'(x)$ with $\psi(x)$ being defined by

$$\psi(x) = \frac{\varphi(x)}{\sin ax}, (x \ne 0), \quad \psi(0) = \lim_{x\to 0} \frac{\varphi(x)}{\sin ax} = 0. \tag{18}$$

Evidently (15) and (16) are respectively equivalent to the following

$$\int_{-\epsilon}^{\epsilon} |f|^n \sin n\theta(x) \varphi(x) dx = O(\epsilon) \int_{-\epsilon}^{\epsilon} |f|^n \cos n\theta(x) dx \tag{19}$$

$$\int_{-\epsilon}^{\epsilon} |f|^n \cos n\theta(x) \varphi(x) dx = O(\epsilon) \int_{-\epsilon}^{\epsilon} |f|^n \sin n\theta(x) dx \tag{20}$$

Replacing $\varphi(x)$ by $\varphi(x) + \varphi(-x)$ and $\varphi(x) - \varphi(-x)$ respectively, one may see that (19) and (20) can be expressed as the integrals over the interval $[0, \epsilon]$, so that in order to prove (15)– (16) it suffices to verify the following relations (with $|O(\epsilon)| < A\epsilon$ for large n)

$$\int_0^{\epsilon} |f|^n \cos n\theta(x) \varphi(x) dx = O(\epsilon) \Phi_n(\epsilon) \tag{21}$$

$$\int_0^{\epsilon} |f|^n \sin n\theta(x) \varphi(x) dx = O(\epsilon) \Phi_n(\epsilon) \tag{22}$$

where $\Phi_n(x)$ is defined by

$$\Phi(t) \equiv \Phi_n(t) := \int_0^t |f(x)|^n \cos n\theta(x) dx, \quad (0 \le t \le \epsilon).$$

Verification of (21) For the integral (21), using integration by parts, we have

$$\int_0^\epsilon |f|^n \cos n\theta(x)\varphi(x)dx = \int_0^\epsilon \varphi(x)d\Phi(x)$$
$$= [\Phi(x)\varphi(x)]_0^\epsilon - \int_0^\epsilon \Phi(x)\varphi'(x)dx = J_1 - J_2.$$

Since $\varphi(0) = \varphi'(0) = 0$, it follows that

$$J_1 = \varphi(\epsilon)\Phi(\epsilon) = O(\epsilon^2)\Phi(\epsilon) = O(\epsilon)\Phi(\epsilon).$$

For the integral J_2, since $\varphi'(x)$ is monotonic in $(0,\epsilon)$ with $\varphi'(0) = 0$, it is seen that an application of Bonnet's mean value theorem gives

$$J_2 = \varphi'(\epsilon)\int_{\delta_1}^\epsilon \Phi(x)dx, \quad (0 < \delta_1 = \delta_1(n) < \epsilon).$$

Notice that $\Phi(x) = \int_0^t \Re(f(x))^n dx = \frac{1}{2}\Delta_n(t)$, so that it possesses the (PB)-property. Consequently J_2 may be estimated as follows

$$\begin{aligned} J_2 &= \varphi'(\epsilon)(\epsilon - \delta_1)\Phi_n(\xi), \quad (\delta_1 < \xi = \xi_n < \epsilon) \\ &= O(\epsilon)O(\epsilon)O(\Phi_n(\epsilon)) = O(\epsilon)\Phi(\epsilon), \end{aligned}$$

where the (UB)-property of $\Phi_n(t)$ ensures that the factors implied in the O's are bounded uniformly for all large n. Hence in conclusion $J_1 = J_2 = O(\epsilon)\Phi_n(\epsilon)$, and (21) is proved.

Verification of (22) Using the function $\psi(x)$ defined by (18), we may rewrite (22) in the form

$$\int_0^\epsilon |f(x)|^n \sin n\theta(x) \sin ax\psi(x)dx = O(\epsilon)\Phi_n(\epsilon) \qquad (22)'$$

Let us introduce the function

$$\Psi(t) \equiv \Psi_n(t) := \int_0^t |f(x)|^n(\cos n\theta(x) - \sin\theta(x)\sin ax)dx, \quad (0 \le t \le \epsilon).$$

Notice that (12) implies that for every $t > 0$,

$$\Psi(t) \sim \int_0^t |f(x)|^n \cos n\theta(x)dx = \Phi_n(t), \quad (n \to \infty),$$

i.e., $\Psi_n(t) = [1 + o(1)]\Phi_n(t) \sim \frac{1}{2}\Delta_n(t)$. Moreover, for $0 \le x \le t \to 0_+$

$$|f(x)|^n|\sin n\theta(x)\sin ax| \le |\sin ax| \to 0,$$

where the limit passage going to zero is independent of n. Thus $\Psi_n(t)$ also enjoy the (UB)-property for small $\epsilon > 0$ and large n.

Just like $\varphi'(x)$, the function $\psi'(x)$ is monotonic in the small interval $(0, \epsilon)$, with $\psi'(x) - \psi'(0)$ being zero for $x = 0$. Thus, using integration by parts and Bonnet's theorem, and recalling the (UB)- property of $\Phi_n(t)$ we obtain

$$\begin{aligned}
&\int_0^\epsilon |f(x)|^n \cos n\theta(x)\psi(x)dx = \int_0^\epsilon \psi(x)d\Phi(x) \\
&= [\Phi_n(x)\psi(x)]_0^\epsilon - \int_0^\epsilon \Phi_n(x)[\psi'(x) - \psi'(0)]dx + \int_0^\epsilon \Phi_n(x)dx\psi'(0)dx \\
&= \Phi_n(\epsilon)\psi(\epsilon) - \psi'(\epsilon)\int_{\delta_2}^\epsilon \Phi_n(x)dx + \psi'(0)\epsilon\Phi_n(\xi_n) \\
&= \Phi_n(\epsilon)\varphi(\epsilon)/\sin a\epsilon - \psi'(\epsilon)(\epsilon - \delta_2)\Phi_n(\xi_n') + \psi'(0)\epsilon\Phi_n(\xi_n) \\
&= \Phi_n(\epsilon)O(\epsilon) - O(1)O(\epsilon)O(\Phi_n(\epsilon)) + O(\epsilon)O(\Phi_n(\epsilon)) \\
&= O(\epsilon)\Phi_n(\epsilon), \quad (0 < \xi_n < \epsilon, \delta_2 < \xi_n' < \epsilon)
\end{aligned} \tag{23}$$

where in the last order term $O(\epsilon)$ the factor is bounded for all large n.

Similary, using integration by parts and Bonnet's theorem again, and making use of the (UB)-property of $\Psi_n(t)$, we get

$$\begin{aligned}
&\int_0^\epsilon |f|^n[\cos n\theta(x) - \sin n\theta(x)\sin ax]\psi(x)dx = \int_0^\epsilon \psi(x)d\Psi_n(x) \\
&= [\Psi_n(x)\psi(x)]_0^\epsilon - \int_0^\epsilon \Psi_n(x)[\psi'(x) - \psi'(0)]dx + \psi'(0)\int_0^\epsilon \Psi_n(x)dx \\
&= O(\epsilon)O(\Psi_n(\epsilon)) = O(\epsilon)\Phi_n(\epsilon)
\end{aligned} \tag{24}$$

Finally, a comparison of (23) with (24) yields (22)$'$, so that (22) is verified. This completes the proof of Theorem 2. □

Remark 1 Worth mentioning is that Lemma 4 is logically equivalent to (3), and that the old open problem posed by Chung [2] may be resolved if (12) could be given an independent proof without any reference to (3) or any probabilistic argument.

Remark 2 As an open question, it is not yet known whether the (UB)- property of $\Delta_n(t)$ can be deduced from the condition (0) or not. We conjecture that the answer may be negative. Perhaps a certain counterexample may be constructed by a special choice of the seequence $\{p_k\}(-\infty < k < \infty)$.

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