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A Theorem on Erdos Sequences *

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Abstract It is shown that the product of the two largest lengths of the longest decreasing and increasing subsequences in any sequence composed of n distinct numbers is not less than n, whilst the sum of the two length is not less that $2\sqrt{n}$. Finally, it is pointed out that a result of Erdös and Szekeres is an immediate consequence of the theorem of this paper, but not vice versa.

Keywords permutation, decreasing sequence, increasing sequence, bicolor graph.

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1. Introduction

Erdös and Szekeres [1] proved that in any sequence composed of $mn + 1$ distinct integers, $u_1, u_2, \cdots, u_{mn+1}$, there exists either a decreasing subsequence longer than m or an increasing subsequence longer than n.

In this paper, we will prove a stronger form of this theorem.

Theorem *Let L^- and L^+ be the lengths of the longest decreasing subsequence and the longest increasing subsequence in a sequence respectively. Then for any sequence composed of n distinct integers, we have*

$$L^- L^+ \geq n, \tag{1.1}$$

and

$$L^- + L^+ \geq 2\sqrt{n}. \tag{1.2}$$

2. Permutations and bicolor graphs

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Without losing generality, suppose that we are given n distinct integers as $N = \{1, 2, \cdots, n\}$. Then each permutation of N corresponds to a permutation matrix [2] as

$$\begin{aligned}[u_1, u_2, \cdots, u_n] &= [1, 2, \cdots, u](a_{ij})_{n\times n} \\ &= [1, 2, \cdots, n] \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{matrix} \\ \\ -\text{Row } u_2 \\ \\ -\text{Row } u_1 \\ \\ -\text{Row } u_n \\ \\ \\ \end{matrix} \end{aligned} \tag{2.1}$$

where $a_{ij} = 1$, if and only if $i = u_j (j = 1, 2, \cdots, n)$, and the other entries of A are all zero.

We take these 1's as the vertices of a graph, and denote these n vertices by $V(i, j)$, where $i = u_j (j = 1, 2, \cdots, n)$.

The fact that $V(i, j)$ and $V(k, l)$ are connected by a blue-edge is symbolized by $V(i, j)BV(k, l)$. Similarly, $V(i, j)RV(k, l)$ symbolizes connecting the two vertices by a red-edge.

We lay down the following rules that

$$V(i, j)BV(k, l) \Longleftrightarrow i < k, j < l (\text{or } i > k, j > l); \tag{2.2}$$

$$V(i, j)RV(k, l) \Longleftrightarrow i < k, j > l (\text{or } i > k, j < l). \tag{2.3}$$

Graphically, an edge going up is colored in red; an edge going down in blue. Thus, we have the bicolor graph corresponding to the given sequence.

$V_1 BV_2 B \cdots B_{V_k}$, a group of vertices connected by blue-edges one by one, is called a blue chain, k in length. Similarly, $V_1 RV_2 R \cdots RV_k$, a red chain.

Evidently, we have

Lemma 1 *A blue (resp., red) chain in the bicolor graph, k in length, corresponds to an increasing (resp., a decreasing) subsequence of the same length in the given sequence.*

3. Blue chain forming method

By C_i we mean a blue chain, and also mean the set of vertices on it. We denote the set of all n vertices by D.

(I) Forming of C_1: Suppose C_1 is $V(i_1, j_1)BV(i_2, j_2)B \cdots BV(i_k, j_k)$, it is formed as follows:

(i) Domain of vertices: D.

(ii) Starting vertex: among the vertices in D, select the vertex $V(i_1, j_1)$ that has the smallest column index j_1, i,e., $V(u_1, 1)$ in the case of C_1, as the starting vertex of C_1,

$$j_1 = \min\{j | V(i, j) \in D\} = 1.$$

(iii) Middle vertices: After selecting $V(i_1, j_1)$, we select $V(i_2, j_2)$ so that

$$j_2 = \min\{j | V(i,j) \in D \text{ and } i > i_1, j > j_1\},$$

namely, $V(i_2, j_2)$ has the smallest column index j_2 among the vertices $V(i,j) \in D$ that satisfy $i > i_1$ and $j > j_1$. Note that $j_1 = 1, i_1 = u_1$, in the case of C_1.

Similarly,

$$\begin{aligned} j_3 &= \min\{j | V(i,j) \in D \text{ and } i > i_2, j > j_2\}, \\ &\cdots \quad \cdots \quad \cdots \\ j_k &= \min\{j | V(i,j) \in D \text{ and } i > i_{k-1}, j > j_{k-1}\}. \end{aligned}$$

(iv) Ending vertex: As ending vertex of $C_1, V(i_k, j_k)$ satisfies

$$\{V(i,j) | V(i,j) \in D \text{ and } i > i_k, j > j_k\} = \emptyset,$$

where $\emptyset$ stands for the empty set.

(II) Forming of $C_i (i > 1)$: Suppose C_i is $V(i_1, j_1) B V(i_2, j_2) B \cdots B V(i_k, j_k)$, it is formed as follows:

(i) Domain of vertices: $D_i = D \backslash (C_1 \cup C_2 \cup \cdots \cup C_{i-1})$, namely, we form C_i among the vertices that do not belong to $C_1, C_2, \cdots, C_{i-1}$.

(ii) Starting vertex: Among the vertices in D_i, we select $V(i_1, j_1)$ that has the smallest column index j_1 as the starting vertex of C_i,

$$j_1 = \min\{j | V(i,j) \in D_i\}. \tag{3.1}$$

(iii) Middle vertices: After selecting $V(i_1, j_1)$, we select $V(i_2, j_2)$ so that

$$j_2 = \min\{j | V(i,j) \in D_i \text{ and } i > i_1, j > j_1\},$$

namely, $V(i_2, j_2)$ has the smallest colume index j_2 among the vertices that do not belong to $C_1, C_2, \cdots, C_{i-1}$, and satisfy $i > i_1, j > j_1$.

Similarly,

$$\begin{aligned} j_3 &= \min\{j | V(i,j) \in D_i \text{ and } i > i_2, j > j_2\}, \\ &\cdots \quad \cdots \quad \cdots \\ j_k &= \min\{j | V(i,j) \in D_i \text{ and } i > i_{k-1}, j > j_{k-1}\}. \end{aligned} \tag{3.2}$$

(iv) Ending vertex: As ending vertex of $C_1, V(i_k, j_k)$ satisfies

$$\{V(i,j) | V(i,j) \in D_i \text{ and } i > i_k, j > j_k\} = \emptyset. \tag{3.3}$$

Continue this procedure untill every vertex in D has been chained.

4. Existence of red chains

Suppose that s blue chains $C_1, C_2, \cdots, C_s$ are formed by the method given above on the permutation matrix. Then, from the blue chain forming process,

Lemma 2 *The set of all $C_i (i = 1, 2, \cdots, s)$ is a partition of the vertex set D[3], i.e.,*

$$C_i \cap C_j = \emptyset (i \neq j), \tag{4.1}$$

$$C_1 \cup C_2 \cup \cdots \cup C_s = D. \tag{4.2}$$

Lemma 3 *Suppose the blue chain* $C_i : V(i_1,j_1)BV(i_2,j_2)B\cdots BV(i_k,j_k)$, *and a vertex* $V(i_0,j_0)$ *satisfy:*

(1)

$$V(i_0,j_0) \notin C_t (t = 1,2,\cdots,i-1); \tag{4.3}$$

(2) For any $V(i_m,j_m) \in C_i$, *we have*

$$V(i_m,j_m)BV(i_0,j_0), \tag{4.4}$$

then $V(i_0,j_0) \in C_i$.

Proof By (4.3),

$$V(i_0,j_0) \in D_i. \tag{4.5}$$

Hence $j_0 \in \{j|V(i,j) \in D_i\}$, compared with (3.1), we have $j_1 \leq j_0$.

If $j_1 = j_0$, then $i_1 = i_0$. This means $V(i_0,j_0) \in C_i$.

Assume that

$$j_1 < j_0. \tag{4.6}$$

By (4.4), $V(i_0,j_0)BV(i_k,j_k)$, and by (2.2), we have

$$i_0 < i_k, j_0 < j_k, \tag{4.7}$$

or

$$i_0 > i_k, j_0 > j_k. \tag{4.8}$$

By (3.3), we know that (4.8) is impossible. Thus, by (4.6) and (4.7), we have $j_1 < j_0 < j_k$. Hence, in the sequence $j_1 < j_2 \cdots < j_k$, there must be a j_t such that j_t is less than j_0 but j_{t+1} is not less than j_0, i.e., $j_{t+1} \geq j_0$. Therefore,

$$j_1 < j_2 < \cdots < j_t < j_0 \leq j_{t+1} < \cdots < j_k, \tag{4.9}$$

and by (3.2),

$$i_1 < i_2 < \cdots < i_t < i_0 \leq i_{t+1} < \cdots < i_k. \tag{4.10}$$

By (4.5), (4.9) and (4.10), we have

$$j_0 \in \{j|V(i,j) \in D_i \text{ and } i > i_t, j > j_t\}, \tag{4.11}$$

but by (3.2), we have

$$j_{t+1} = \min\{j|V(i,j) \in D_i \text{ and } i > i_t, j > j_t\}. \tag{4.12}$$

Compared (4.12) with (4.11), we obtain

$$j_0 \geq j_{t+1}. \tag{4.13}$$

By (4.13) and (4.9), we have $j_0 = j_{t+1}$, hence $i_0 = i_{t+1}$.

Thus, from $V(i_{t+1}, j_{t+1}) \in C_i$, we obtain $V(i_0, j_0) \in C_i$. This completes the proof. □

If C is a chain, by $|C|$ we denote the length of C.

Lemma 4 *If s blue chains are formed in the graph by the method we give, then there exists at least a red chain C' with $|C'| = s$.*

Proof Suppose that the s blue chains are $C_1, C_2, \cdots, C_s$. We take any vertex $V_s \in C_s(|C_s| \geq 1)$, then there is at least a vertex $V_{s-1} \in C_{s-1}$ such that $V_{s-1}RV_s$. Otherwise, we would have VBV_s, for any $V \in C_{s-1}$, and this would lead to $V_s \in C_{s-1}$ by Lemma 3. A contradiction.

In the similar way, there are $V_{s-2} \in C_{s-2}, \cdots, V_2 \in C_2, V_1 \in C_1$ such that $V_1RV_2R\cdots RV_s$. We denote this red chain by C', and we have $|C'| = s$.

5. Proof of the Theroem

Suppose that in the graph we have s blue chains $C_1, C_2, \cdots, C_s$. Let $|C_i| = c_i (i = 1, 2, \cdots, s)$. Obviously, $c_i \geq 1$. By Lemma 4, we have at least a red chain C', and $|C'| = s$. Therefore, by Lemma 1, we have

$$L^{-1} \geq |C'| = s. \tag{5.1}$$

By Lemma 1 and 2, we obtain

$$L^{+} = \max\{c_1, c_2, \cdots, c_s\} \geq \frac{1}{s}\sum_{i=1}^{s} c_i = \frac{n}{s}. \tag{5.2}$$

By (5.1) and (5.2), we have $L^- L^+ \geq n$ and $L^- + L^+ \geq 2\sqrt{n}$. These complete the proof of the theorem.

6. Some remarks

A. For a sequence composed of $mn + 1$ distinct integers, according to the theorem in this paper, we have $L^- L^+ \geq mn + 1$. Thus, we easily obtain $L^- > m$ or/and $L^+ > n$. This proves the theorem of Erdös and Szekeres.

B. From the argument above, we notice that the longest decreasing subsequence and the longest increasing subsequence we choose have and olny have one common item.

C. When $c_1 = c_2 = \cdots = c_s$ or $s = 1$, the equality (1.1) holds; when $c_1 = c_2 = \cdots = c_s = s$, the equality (1.2) holds.

References

[1] P.Erdös & G.Szekeres, *A combinatorial problem in geometry*, Composition Math., 2, 1935, 463–470.

[2] M.Hall, *The Theory of Groups* , Macmillan Company, New York, 1959.

[3] L.Comtet, *Advanced Combinatorics*, Reidel publishing Company, 1974.