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A Remark on Multivariate Polynomial Interpolations *

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Abstract C.deBoor and A.Ron in [1] proposed a map $\Theta \to \Pi_\Theta$ which associates each finite set Θ in complex s-space with a polynomial space Π_Θ from which interpolation to arbitrary data given at the points in Θ is possible and uniquely. In this paper we describe the constructing methods of some other polynomial spaces Q from which interpolation at Θ is uniquely possible. This method is utilized to construct some nonconforming finite elements (cf.[3,4]) with good convergence.

Keywords Interpolation, finite element.

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1. Introduction

The recent work of Z.C. Shi and S.C. Chen (see, e.g., [2] and references therein) serves as a good example for the explanation of nonconforming finite elements. They showed that the nine parameters of quasi-conforming plate element are the point valuation of shape function and first derivatives with some disturbance of $O(h^2)$, and presented the explicit expressions of those nine parameters. Their conclusions result in the fact that most of nonconforming plate elements can be explained as the polynomial interpolations as well as the conforming elements. From this new point of view, the construction of nonconforming finite element is also the interpolation problem of polynomials with some complicated interpolation conditions (which are viewed as the sufficient conditions for convergence of finite elements) and the subspaces of polynomials from which interpolation is uniquely possible.

The generalization of univariate polynomial interpolation to the multivariate context is made difficult by the fact that it has to be decided just which of the many of its nice properties to preserve, since it is impossible to preserve them all. In [1], C.deBoor and A.Ron took a different task. Given any finite set $\Theta \subset R'$, they determined a corresponding polynomial space Q from which interpolation to function values and derivatives at Θ is "correct", i.e., is possible and uniquely so. Such a basic idea can be used to construct nonconforming finite elements which have good convergent properties.

This paper generalizes the conclusions of C.deBoor and A.Ron in [1] The generalization can be used for the construction of nonconforming finite elements in a better way.

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The following notation and terminology are used throughout. The collection of all polynomials on C' is denoted by π; π_k denotes the collection of all those polynomials of total degree $\le k$, i.e., $\pi_k := \text{span}(()^\alpha), |\alpha| \le k$, with $()^\alpha : x \to x^\alpha$. For any $p \in \pi$, we denote by $p(D)$ the corresponding constant coefficient differential operator; in particular, $D^\alpha := \prod_j (D_j)^{\alpha(j)}$, with D_j a differentiation with respect to the jth argument. We make good use of the representation of the linear functional $[\theta]p(D)$ on π as $q \to p^* E^\theta q = q^*(e_\theta p)$, with

$$q^* p := (q(D)p)(0) = \sum_\alpha D^\alpha q(0) D^\alpha p(0)/a!,$$

with $e_\theta : x \to \exp(\langle\theta, x\rangle)$, with E the shift, i.e., $E^\theta f = f(\cdot + \theta)$, and with $[\theta]f := f(\theta)$.

2. Baic interpolation problems

Let H and Λ be finite dimensional linear subspaces of a linear space X (over C or R) and its dual X', respectively. Abstractly, interpolation from H can be described as the task of finding, for given $f \in X$, an $h \in H$ for which $\lambda h = \lambda f$ for all λ in Λ. We call Λ the (space of) interpolation conditions for this particular interpolation problem (H, Λ). We call the problem correct if there is, for each f, exactly one solution h form H.

A space Λ of linear functions is total for H if the only $h \in H$ for which $\lambda h = 0$ for all $\lambda \in \Lambda$ is $h \equiv 0$. Then we have the following basic lemma.

Lemma 2.1 *Let H and Λ be finite demensional linear subspaces of a linear space X and its dual, respectively. Then the following are equivalent:*

(a) The interpolation problem (H, Λ) given by H and Λ is correct.

(b) With $(\lambda_j)_1^*$, any basis for Λ, the linear map $T_H : h \to (\lambda_j h)_1^*$ is one-one and onto.

(c) Λ is minimally total for H.

(d) Λ can be used to represent the dual H' of H in the sense that the map $F_\Lambda : \lambda \to \lambda|_H$ is one-one and onto.

For polynomial interpolation problem we are interested in the case where X is the space, A_0, of all functions analytic at the origin with the topology of formal power series Concretely, we are interested in using linear functionals of the form

$$\bar{p}^* : f \to (\bar{p}(D)f)(0) = \sum_\alpha \overline{D^\alpha p(0)} \cdot D^\alpha f(0)/\alpha! \qquad (2.1)$$

with $p \in \pi$. These are continuous linear functionals on A_0 and even on some $C^k(0)$. The map $p \to \bar{p}^*$ is skew linear and one-to-one, hence provides a skew-linear embedding of π in the dual of A_0.

We will consider interpolation by polynomials using interpolation conditions of the form $[\theta]p(D)$ with $[\theta]$ the linear functional of point evaluation of θ and p a polynomial. More precisely, let $\Theta_i (i = 1, 2, \cdots, n) \subset C'$ be n finite sets and associate each point θ in Θ_i with a polynomial p_θ. We want to interpolate from some polynomial space Q, using the interpolation conditions

$$\Lambda = \Lambda(\Theta_1, \cdots, \Theta_n) = \text{span}\{\sum_{\theta \in \Theta_i} [\theta] p_\theta(D) : i = 1, 2, \cdots, n\} \qquad (2.2)$$

For the analysis of this problem, observe that, in term of (2.1), for any $\Theta = \Theta_i$ and $q \in \pi$

$$(\sum_{\theta\in\Theta}[\theta]p_\theta(D))q = \sum_{\theta\in\Theta}[\theta]p_\theta(D)q = \sum_{\theta\in\Theta} q^*(e_\theta p_\theta) = q^* \sum_{\theta\in\Theta}(e_\theta p_\theta) \qquad (2.3)$$

This implies that our interpolation problem, as specified by Q and $\Lambda = \Lambda(\Theta_1, \cdots \Theta_n)$ is correct if and only if the dual problem of interpolation from $H := \text{span}\{\sum_{\theta\in\Theta_i}(e_\theta p_\theta) : i = 1, \cdots, n\}$ with interpolation condition $\overline{Q}^*$ is correct. Now H is a subspace of A_0 and Q is a subspace of polynomial to be determined. For interpolation problem $(H, \overline{Q}^*)Q^*$ can be used to represent the dual H' of H and conversely so, in terms of (d) of Lemma 2.1. Hence finding Q for original interpolation problem (Q, Λ) is equivalent to determining H', the dual of H.

3. Methods for constructing H'

For a function (or polynomial) f analytic at the origin, let $f_\downarrow$ be the homogencous polynomial of degree j for which

$$f(x) = f_\downarrow(x) + o(\|x\|^j) \text{ as } x \to 0 \qquad (3.1)$$

Consequently, with $j := \deg f_\downarrow$

$$f_\downarrow = \lim_{t\to 0} f(t\cdot)/t^j.$$

For any subspace H of A_0, we first consider the polynomial space

$$H_\downarrow := \text{span}\{f_\downarrow : f \in H\} \qquad (3.2)$$

Proposition 3.1 *For any finite-dimensional linear subspace H of A_0, the linear space $\overline{H}_\downarrow^* = \{\bar{p}^* : p \in H_\downarrow\}$ can be used to represent dual of H.*

Proof For any $f \in H\backslash 0, p := f_\downarrow \in H_\downarrow$ and $\bar{p}^* f = \bar{p}^* p > 0$. This implies that the only $f \in H$ with $p^* f = 0$ for all $p \in \overline{H}_\downarrow$ is $f \equiv 0$, i.e., $\overline{H}_\downarrow^*$ is total for H. On the other hand, since $\dim \overline{H}_\downarrow^* = \dim \overline{H}_\downarrow = \dim H$, on proper subspace of $\overline{H}_\downarrow^*$ could be total for H. That is, $\overline{H}_\downarrow^*$ is minimally total for H and then proposition is valid in terms of Lemma 2.1.

Let $h_1, \cdots, h_n \in H$ be a basis of H with $\dim H = n$(or in general $H = \text{span}_{1\le i\le m}\{h_i\}$), then for any $h \in H$, there exist $c_1, \cdots, c_{n-1}$ and c_n such that

$$h = c_1 h_1 + c_2 h_2 + \cdots c_n h_n = \sum_\alpha L_\alpha(c_1, \cdots, c_n) \cdot x^\alpha \qquad (3.3)$$

where $L_\alpha(c_1, \cdots, c_n)$ is the linear form of $c_1, \cdots, c_n$.

Proposition 3.2 *There exist an integer m_0 such that*

$$H_\downarrow := \bigoplus_{j=}^{m_0}\{\sum_{|\alpha|=j} L_\alpha(c_1, \cdots, c_n)x^\alpha : L_\alpha(c_1, \cdots, c_n) = 0 \text{ with } |\alpha| < j\}$$

Proof We just note that the element of H can be identified by the first nonzero homogeneous polynomial in its Taylor expansion and H is finite dimensional subspace of A_0.

Let m_0 be the integer in Proposition 3.2. For any given integer $J \geq m_0$ define $H(J) = \text{span}\{T_J f : f \in H\}$ with $T_J f$ the Taylor polynomial of degree $\leq J$ for f at the origin, i.e.,

$$T_J f = \sum_{|\alpha| \leq J} \frac{x^\alpha}{a!} D^\alpha f(0) \tag{3.4}$$

For any $f \neq 0$ in H we have $\overline{T_J f^*} \cdot f = \overline{T_J f^*} T_J f > 0$. This implies that only $f \in H$ with $p^* f = 0$ for all $p \in \overline{H(J)}$ is $f \equiv 0$. Hence we can conclude the following

Proposition 3.3 *The linear space $\overline{(H(J))}^*$ is minimally total for H and*

$$H(J) = \bigoplus_{j=0}^{m_0} \{ \sum_{|\alpha|=j}^{J} L_\alpha(c_1, \cdots, c_n) x^\alpha : L_\alpha(c_1, \cdots, c_n) = 0 \text{ with } |\alpha| < j \}$$

Denote $L_j(J) = \{\sum_{|\alpha|=j}^{j} L_\alpha(c_1, \cdots, c_n) x^\alpha : L_\alpha(c_1, \cdots, c_n) = 0 \text{ with } |\alpha| < j\}$ and $H_\downarrow(J) = \oplus_{j=0}^{m_0 - 1} L_j(J)_\downarrow \oplus L_{m_0}(J)$. We are now in a position to state following important proposition.

Proposition 3.4 *For any given integer $J \geq m_0$, we have the following statements:*

(a) $\overline{H_\downarrow(J)}^*$ is also minimally total for H and $\dim H_\downarrow(J) = \dim H$. In particular, $H_\downarrow(m_0) = H_\downarrow$.

(b) $\dim(H_\downarrow(m_0) \cap \pi_k) \geq \dim(H_\downarrow(m_0 + 1) \cap \pi_k) \geq \dim(H_\downarrow(m_0 + 2) \cap \pi_k) \geq \cdots \forall k$,

(c) The linear space $H_\downarrow(J)^*$ can be used to represent the dual of H.

Proposition 3.4 provides the following conclusions.

Theorem 3.5 *Let $H = \text{span}\{\sum_{\theta \in \Theta_i} e_\theta p_\theta : i = 1, 2, \cdots, n\}$. Then $\overline{H}_\downarrow$ with $J \geq m_0$(m_0 can be determined in terms of proposition 3.2) is a polynomial space from which interpolation with interpolation condetions $\Lambda(\Theta_1, \cdots, \Theta_n) = \text{span}\{\sum_{\theta \in \Theta_i} [\theta] p_\theta(D) : i = 1, \cdots, n\}$ is correct.*

4. Some examples

Example 4.1 As a simple illustration, consider $s = 1$. For $\Theta_1 = \{x_1\}, \Lambda = \text{span}\{[x_0] + [x_0] \cdot D, [x_1]\}$. Then the corresponding $H = \{c_1(1 + x)e^{x_0 x} + c_2 e^{x_1 x} : c_1, c_2 \in R\}$

(a) For $x_1 \neq 1 + x_0, H_\downarrow = \{c_1 + c_2 x : c_1, c_2 \in R\}, m_0 = 1$. For $J \geq m_0$

$$\begin{aligned} H_\downarrow(J) &= \{c_1 + c_2((x_0 + 1 - x_1)x + \frac{1}{2}(x_0(x_0 + 2) - x_1^2)x^2 + \frac{1}{3!}(x_0^2(x_0 + 3) \\ &\quad - x_1^3)x^3 + \cdots + \frac{1}{J!}(x_0^{J-1}(x_0 + J) - x_1^J)x^J : c_1, c_2 \in R\} \end{aligned}$$

(b) For $x_1 = 1 + x_0, H_\downarrow = \{c_1 + c_2 x^2 : c_1, c_2 \in R\}, m_0 = 2$. For $J \geq m_0$

$$\begin{aligned} H_\downarrow(J) &= \{c_1 + c_2(-\frac{1}{2}x^2 + \frac{1}{3!}((x_1 - 1)^2(x_1 + 2) - x_1^3)x^3 \\ &\quad + \cdots + \frac{1}{J!}((x_1 - 1)^{J-1}(x_1 + J - 1) - x_1^J)x^J) : c_1, c_2 \in R\} \end{aligned}$$

By Theorem 3.5, we have that for any given data α_1 and α_2 there exists only one polynomial $p(x) \in H_\downarrow(J)$ such that

$$p(x_0) + p'(x_0) = \alpha_1 \quad \text{and} \quad p(x_1) = \alpha_2$$

Example 4.2 Let $s = 2$. Let K be the triangle with vertices at $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ in R^2. The mine parameters of Zienkiewicz's finite element is the following.

$$v(x_i, y_i), v_x(x_i, y_i), v_y(x_i, y_i), i = 1, 2, 3. \tag{4.1}$$

In [5], the shape function space (or interpolation polynomial space) was given. Here we can give another nine-dimensional polynomial space with can be viewed as a new shape function space of Zienkiewicz's finite plate elements. For simplicity we just take $(x_1, y_1) = (0, 0), (x_2, y_2) = (1, 0)$, and $(x_3, y_3) = (0, 1)$ as an example. For the general case, please refer to [3]. Now we have

$$\Lambda = \text{span}\{[x_i, y_i], [x_i, y_i] \cdot p_1(D), [x_i, y_i] \cdot p_2(D), i = 1, 2, 3\}$$

with $p_1(x, y) = x$ and $p_2(x, y) = y$, then the corresponding

$$H = \text{span}\{e^{x_i x + y_i y}, x e^{x_i x + y_i y}, y e^{x_i x + y_i y}, i = 1, 2, 3\}$$

where $(x_1, y_1) = (0, 0), (x_2, y_2) = (1, 0)$ and $(x_3, y_3) = (0, 1)$.
By Theorem 3.5, we can prove that

$$H_\downarrow \cong \pi \oplus \{c_1 x^3 + c_2(x^2 y - x y^2) + c_3 y^3 : c_1, c_2, c_3 \in R\}, m_0 = 3$$

Moreover, for example

$$\begin{aligned} H_\downarrow(4) \quad &\cong \quad \pi_2 \oplus \{c_1(x^3 + \frac{1}{2}x^4) + c_2(x^2 y - x y^2 + \frac{1}{3}x^3 y - \frac{1}{3}x y^3) \\ &\quad + c_3(y^3 + \frac{1}{2}y^4) : c_1, c_2, c_3 \in R\} \end{aligned}$$

Then we have that $H_\downarrow$ and $H_\downarrow(4)$ are shape function space of Zienkewicz's finite element over K.

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