Theorem 5 *If $P(z)$ is defined by (4.1), then*

$$\sup_{|y|\leq\frac{1}{2}}\{\frac{1}{\pi}\int_{\pi\sigma}^{\infty}\frac{1-\cos ty}{t^{2r}|P(it)|^2}dt\}^{1/2}\leq \bar{d}_\sigma(W_1^r(P),L^2)$$

$$\leq E(W_1^r(P),B_{\pi\sigma}^2,L^2)\leq \sup_{y\in\mathbb{R}}\{\frac{1}{\pi}\int_{\pi\sigma}^{\infty}\frac{(1-\cos ty)dt}{t^{2r}|P(it)|^2}\}^{1/2}.$$

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$L(\mathbb{R})$ 中卷积函数类在 $L_2(\mathbb{R})$ 中的平均 $\sigma-K$ 宽度

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摘　要

本文研究了 $L(\mathbb{R})$ 中的卷积函数类在 $L_2(\mathbb{R})$ 中的平均 $\sigma-K$ 宽度且得到一些精确结果．同时，对非卷积情形，得到一些渐近结果．

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Average $\sigma - K$ Width of Convolution Function Class of $L(\mathbb{R})$ in $L_2(\mathbb{R})$. *

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Abstract In this paper, we study the average $\sigma - K$ widths of the convolution classes of functions of $L(\mathbb{R})$ in $L^2(\mathbb{R})$ and obtain some exact results. Meanwhile, for the non-convolution case, we also obtain some asymptotic results.

Keywords average $\sigma - K$ width, optmal subspace, average dimension, convolution class.

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1. Introduction

Let X be a normed linear space with norm $\|\cdot\|$ and M a subspace of X. For a subset $\mathcal{M}$ of X, set

$$E(\mathcal{M}, M, X) =: \sup_{f\in\mathcal{M}} e(f, M, X),$$

where $e(f, M, X) =: \inf\{\|f-g\| : g \in M\}$ for any $f \in \mathcal{M}$.

For $p \in [1, +\infty)$, let A be a linear subspace of infinite dimension of $L^p =: L^p(\mathbb{R})$ ($\mathbb{R}$ is the real line), and set $B(A) = \{f \in A : \|f\|_p \le 1\}$. For any $\epsilon > 0$ and $a > 0$, set $k_\epsilon(a, L, L^p) =: \min\{m :$ there exists a subspace M of dimension m of $L^p(I_a)$ such that $E((BA)(I_a), M, L^p(I_a)) \le \epsilon\}$, where $I_a =: [-a, a], \|\cdot\|_p$ denotes the usual $L^p(\mathbb{R})$-norm defined by $\|f\|_p = \{\int_{\mathbb{R}} |f(t)|^p dt\}^{1/p}$ for any $f \in L^p$ and $(BA)(I_a) =: \{f|_{I_a} : f \in B(A)\}$. If there exists a positive real number $\sigma > 0$ such that

$$\overline{\dim}(A, L^p) =: \lim_{\epsilon\to 0} \varliminf_{a\to\infty} \frac{k_\epsilon(a, A, L^p)}{2a} = \sigma,$$

then, the subspace A is called to be of average σ-dimension.

Let $\mathcal{M}$ be a subset of L^p. The quantity

$$\overline{d_\sigma}(\mathcal{M}, L^p) =: \inf\{E(\mathcal{M}, A, L^p) : \overline{\dim}(A, L^p) \le \sigma\}$$

is called the Kolmogorov average σ-width of $\mathcal{M}$ in L^p (shortly, average $\sigma - K$ width). If there exists a subspace A^*_σ of L^p of average dimension $\le \sigma$ such that $\overline{d_\sigma}(\mathcal{M}, L^p) = E(\mathcal{M}, A^*_\sigma, L^p)$, then A^*_σ is called an optimal subspace realizing $\overline{d_\sigma}(\mathcal{M}, L^p)$.

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Let $G \in L^1$. Set

$$B_1(G) = \{G * \varphi : \|\varphi\|_1 \le 1\}, \tag{1.1}$$

where $G * \varphi$ denotes the convolution transform with kernel G of the function φ, defined by

$$(G * \varphi)(x) =: \int_{\mathbb{R}} G(x-t)\varphi(t)dt. \tag{1.2}$$

Let $r \in Z_+ =: \{1, 2, \cdots\}$. For any $F \in C(\mathbb{R})$, set

$$W_1^r F =: \{f \in L^2 : \text{ there exists } h \in B(L^1) \text{ such that } (it)^r f^\wedge(t) = F(t)h^\wedge(t)\}, \tag{1.3}$$

where the Fourier transform $f^\wedge$ of $f \in L^1$ is defined by $f^\wedge(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t)e^{-itx}dt$, and of L^2 by the limit in $L^2(\mathbb{R})$-norm of $\frac{1}{\sqrt{2\pi}} \int_{-\rho}^{\rho} f(t)e^{-itx}dt, \rho \to +\infty$. When $F(x) = 1, W_1^r F = W_1^r(R)$ is the usual Sobolev class.

Since the concept of the average widths was proposed by Tikhomirov [15], the average $\sigma - K$ width problems of many smooth function classes of $L^p(\mathbb{R})$ in $L^q(\mathbb{R})(1 \le q \le p \le \infty)$ have been studied and many exact results have been obtained (see cf. [2,5,9]). The purpose of this paper is to study the average $\sigma - K$ width problems of the classes $B_1(G)$ and $W_1^r F$ in L^2 for some proper functions $G, F \in C(\mathbb{R})$, respectively. We obtain two main theorem as follows.

Theorem 1 *Let $G \in L^1$. If there exists a constant C such that $|G(x)| \le \frac{C}{1+x^2}$ for any $x \in \mathbb{R}$, then*

$$\overline{d_\sigma}(B_1(G), L^2) = E(B_1(G), B^2_{E(\sigma)}, L^2) = \|G^\wedge\|_{L^2(\mathbb{R} - E(\sigma))}, \tag{1.4}$$

where $\|f\|_{L^2(E)} =: \{\int_E |f(t)|^2 dt\}^{1/2}, B_E^2 =: \{f \in L^2 : \ supp\ f^\wedge \subset E\}$, for the Lebesgue measurable subset E of $\mathbb{R}$, while $E(\sigma)$ is a subset of $\mathbb{R}$, defined by

$$\sup\{\|G^\wedge\|_{L^2(E)} : E \subset \mathbb{R},\ \mathrm{mes}E = 2\pi\sigma\} = \|G^\wedge\|_{L^2(E(\sigma))}.$$

Theorem 2 *Let $r \in Z_+$. If $F(x)$ in $C(\mathbb{R}), F(0) \ne 0$, has a bounded and continuous second derivative, and $|F(x)|$ is even on $\mathbb{R}$ and nonincreasing on $(0, \infty)$, then*

$$\begin{aligned} &\sup_{|y|<\frac{1}{\sigma}} \{\frac{1}{\pi} \int_{\pi\sigma}^{\infty} \frac{|F(t)|^2}{t^{2r}}(1 - \cos ty)dt\}^{1/2} \le \overline{d_\sigma}(W_1^r F, L^2) \\ &\le E(W_1^r F, B^2_{\pi\sigma}, L^2) \le \sup_{y \in \mathbb{R}} \{\frac{1}{\pi} \int_{\pi\sigma}^{\infty} \frac{|F(t)|^2}{t^{2r}}(1 - \cos ty)dt\}^{1/2}, \end{aligned} \tag{1.5}$$

where $B^2_{\pi\sigma} =: B^2_{[-\pi\sigma, \pi\sigma]}$ denotes the set of all entire functions which $f \in L^2$ when restricted on $\mathbb{R}$.

2. Proof of Theroem 1

To prove Theorem 1, we need

Lemma 2.1 (cf. [4]) *If E is a Lebesque measurable subset of $I\!R$, then*

$$\overline{\dim}(B_E^2, L^2) = \frac{\mathrm{mes}E}{2\pi}. \tag{2.1}$$

Specially,

$$\overline{\dim}(B_{\pi\sigma}^2, L^2) = \sigma, \tag{2.2}$$

for any $\sigma > 0$.

Lemma 2.2 *Let $G \in L^1 \cap L^2$. Then,*

$$E(B_1(G), B_E^2, L^2) \le \|G^\wedge\|_{L^2(I\!R - E)}. \tag{2.3}$$

Proof For each $f \in B_1(G)$, let $g(f) \in B_E^2$ defined by

$$(g(f))^\wedge(x) = \begin{cases} f^\wedge(x), & x \in E, \\ 0, & x \in I\!R - E. \end{cases} \tag{2.4}$$

For any $f(x) = G * \varphi \in B_1(G)$, since $|\varphi^\wedge(x)| \le \frac{1}{\sqrt{2\pi}}$ and $f^\wedge(t) = \sqrt{2\pi}G^\wedge(t)\varphi^\wedge(t)$, then by Plancherel theroy (cf. [1]), we have

$$\begin{aligned} \|f - g(f)\|_2 &= \{\int_{I\!R-E} |f^\wedge(t)|^2 dt\}^{1/2} = \sqrt{2\pi}\{\int_{I\!R-E} |G^\wedge(x)\varphi^\wedge(x)|^2 dt\}^{1/2} \\ &\le \sqrt{2\pi} \max_{x \in I\!R} |\varphi^\wedge(x)| \|G^\wedge\|_{L^2(I\!R-E)} \le \|G^\wedge\|_{L^2(I\!R-E)}. \end{aligned} \tag{2.5}$$

By (2.5), we get Lemma 2.2.

Proof of Theorem 1 Firstyly, we prove the inequality,

$$\overline{d_\sigma}(B_1(G), L^2) \ge \|G^\wedge\|_{L^2(I\!R - E(\sigma))}. \tag{2.6}$$

Set $\Delta(G) = \{G(\cdot - t) : t \in I\!R\}$. If A be a subspace of L^2 of average dimension $\le \sigma$, then, by the definition of average dimension, for any $N > 0$, there exists a subspace M of dimension $k(N)$ of $L^2(I_N)$ such that

$$E((BA)(I_N), M, L^2(I_N)) < \epsilon,$$

where $k(N) =: k_\epsilon(N, A, L^2)$.

For each $t \in I\!R$, let $f_t(x) =: G(x - t)$. Then for any $g \in A$ we have

$$e(f_t, M, L^2(I_N)) \le \|f_t - g\|_2 + \epsilon\|g\|_2 \le (1 + \epsilon)\|f_t - g\|_2 + \epsilon\|f_t\|_2. \tag{2.7}$$

Set $P_\eta(x) = \frac{\eta}{\pi(x^2+\eta^2)}, \eta > 0$. Then $(P_\eta)^\wedge(x) = e^{-\eta|x|}, \int_{I\!R} P_\eta(x)dx = 1$ and $\|P_\eta * \varphi - \varphi\|_2 \to 0$ as $\eta \to 0^+$ for any $\varphi \in L^2(I\!R)$ (see cf. [13]).

For any $t \in I\!R$, notice that

$$e(f_t * P_\eta, A, L^2) \le E(B_1(G), A, L^2), \tag{2.8}$$

we have

$$e(f_t, A, L^2) = \lim_{\eta\to 0^+} e(f_t * P_\eta, A, L^2) \le E(B_1(G), A, L^2). \tag{2.9}$$

Hence, by (2.7), (2.8) and (2.9), we get

$$e(f_t, M, L^2(I_N)) \le (1+\epsilon)E(B_1(G), A, L^2) + \epsilon\|G\|_2, \tag{2.10}$$

for any $t \in \mathbb{R}$.

Set $F(x) = \sum_{j\in Z} G(2Nx - N + 2jN), x \in [0,1]$. Then

$$F(x) \sim \sum_{k\in Z} F^\wedge(k)e^{2k\pi ix} = \frac{\sqrt{2\pi}}{2N}\sum_{k\in Z} G^\wedge(\frac{k\pi}{N})e^{-ik\pi}e^{2k\pi ix}. \tag{2.11}$$

Thus, we have

$$e(f_t, M, L^2(I_N)) \ge e(\sum_{j\in Z} f_t(\cdot + 2jN), M, L^2(I_N)) - \eta_t(N) = E_N(\frac{t}{2N}) - \eta_t(N), \tag{2.12}$$

where $\eta_t(N) =: \|\sum_{j\ne 0} f_t(\cdot + 2jN)\|_{L^2(I_N)}$ and $E_N(t) =: (2N)^{\frac{1}{2}}e(F(\cdot + t), M(N), L^2[0,1])$, while $M(N) =: \{g(2Nx - N) : g \in M\}$.

By Pinkus [10], we see

$$\max_{0\le t\le 1} E_N^2(t) \ge \int_0^1 E_N^2(t)dt \ge \sum_{k\in Z-\Box_N} |G^\wedge(\frac{k\pi}{N})|^2 \cdot \frac{\pi}{N}, \tag{2.13}$$

where $\Box_N$ is a subset of Z with card $\Box_N = k(N)$ such that

$$|G^\wedge(\frac{k\pi}{N})| \le |G^\wedge(\frac{k'\pi}{N})|, \forall k' \in \Box_N, k \in Z\backslash\Box_N.$$

By a proper discussion, it is easy to verify that

$$\lim_{N\to\infty}\sum_{k\in Z\backslash\Box_N} G^\wedge(\frac{k\pi}{N})|^2\frac{\pi}{N} = \{\int_{\mathbb{R}-E_\sigma}|G^\wedge(x)|^2dx\}^{1/2}. \tag{2.14}$$

Hence, by (2.10) and (2.12), we have

$$\begin{aligned}\int_0^1 E_N^2(t)dt &= \int_\delta^{1-\delta} E_N^2(t)dt + \int_0^\delta E_N^2(t)dt + \int_{1-\delta}^1 E_N^2(t)dt\\ &\le \max_{\delta\le t\le 1-\delta} E_N^2(t) + \int_0^\delta E_N^2(t)dt + \int_{1-\delta}^1 E_N^2(t)dt\\ &\le \max_{2\delta N\le t\le(1-\delta)2N}(e(f_t, M, L^2(I_N)) + \eta_t(N)) + \int_{[0,1]\backslash[\delta,1-\delta]} E_N^2(t)dt\\ &\le \max_{2\delta N\le t\le(1-\delta)2N}\eta_t(N) + (1+\epsilon)E(B_1(G), A, L^2)\\ &\quad +\epsilon\|G\|_2 + \int_{[0,1]\backslash[\delta,1-\delta]} E_N^2(t)dt\end{aligned} \tag{2.15}$$

for any $0 < \delta < \frac{1}{2}$. Since $|G(x)| \le \frac{C}{1+x^2}$, then it is easy to see that

$$\lim_{N\to\infty} \max_{2\delta N \le t \le (1-\delta)2N} \eta_t(N) = 0, \tag{2.16}$$

and there is a constant C_1 such that

$$E_N^2(t) \le (2N)\|F(\cdot+t)\|^2_{L^2[0,1]} = \sum_{k\in Z} |G^\wedge(\frac{k\pi}{N})|^2 \cdot \frac{\pi}{N} \le C_1. \tag{2.17}$$

By (2.14)–(2.17), we have

$$\|G^\wedge\|_{L^2(\mathbb{R}-E_\sigma)} \le (1+\epsilon)E(B_1(G), A, L^2) + \epsilon\|G\|_2 + 2C_1\delta \tag{2.18}$$

Letting $\delta \to 0^+, \epsilon \to 0^+$ in (2.18), we have

$$\|G^\wedge\|_{L^2(\mathbb{R}-E_\sigma)} \le E(B_1(G), A, L^2). \tag{2.19}$$

According to the definition of average $\sigma - K$ width, from (2.19) we see (2.6).

3. Proof of Theorem 2

As in [1], for any $c \in (0,1)$, let $\eta_c(t)$ be an even function which satisfies the following conditions:

(i) $\eta_c(t) = 1, |t| > c\pi$,

(ii) $\eta_c(t)$ has a continuous second derivative, and $0 \le \eta_c(t) \le 1, t \in \mathbb{R}$,

(iii) The rations $\frac{\eta_c''(t)}{t^r}, \frac{\eta_c'(t)}{t^{r+1}}, \frac{\eta_c(t)}{t^{r+2}}$ does not become infinite.

Let F be a continuous and bounded function. Set

$$K(x) =: \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\eta_c(t)}{(it)^r} F(t) e^{itx} dt. \tag{3.1}$$

If $F(0) \ne 0$, then for any $f \in W_1^r F$ there exists $h \in L^1, \int_{\mathbb{R}} h(t)dt = 0$, such that

$$f^\wedge(t) = (it)^{-r} F(t) h^\wedge(t). \tag{3.2}$$

Hence, it is easy to see that

$$\sqrt{2\pi} f(x) - (K * h)(x) \in B^2_{c\pi}. \tag{3.3}$$

Set

$$B_1^0(K) = \{K * h : f^\wedge(x) = (it)^{-r} F(t) h^\wedge(x), f \in W_1^r F\}.$$

Lemma 3.1 *If $F, F(0) \ne 0$, is a continuous and bounded function, then for any $c \in (0,\sigma)$,*

$$E(W^r F, B_E^2, L^2) \le \sup_{y\in\mathbb{R}} \{\frac{1}{2\pi} \int_{\mathbb{R}-E} \frac{|F(t)|^2}{t^{2r}} (1-\cos ty) dt\}^{1/2}, \tag{3.4}$$

where $E \supset (0, c\pi)$.

Proof By the dual theorem of best approximation (cf.[11]) and (3.3), it is easy to see that

$$\begin{aligned}
\sqrt{2\pi}E(W_1^rF,B_E^2,L^2) &= E(B_1^0(K),B_E^2,L^2)\\
&=\sup\{\int_{\mathbb{R}} f(t)g(t)dt : f\in B_1^0(K), g\in B(L^2), g\perp B_E^2\}\\
&=\sup\{e_1(\overline{K}*g)_C; g\in B(L^2) \text{ and } g\perp B_E^2\}\\
&\le \frac{1}{2}\sup_{\substack{g\in L^2\\ g\perp B_E^2}}\sup_y \|(\overline{K}*g)(\cdot+\frac{y}{2})-(\overline{K}*g)(\cdot-\frac{y}{2})\|_C\\
&\le \frac{1}{2}\sup_y e(\overline{K}(\cdot+\frac{y}{2})-\overline{K}(\cdot-\frac{y}{2}),B_E^2,L^2)\|g\|_2\\
&\le \sup_y\{\int_{\mathbb{R}-E}\frac{|F(t)|^2}{t^{2r}}\eta_c^2(t)(1-\cos ty)dt\}^{1/2},
\end{aligned} \tag{3.5}$$

where $e_1(f)_C =: \inf_a\sup_t|f(t)-a|$, $\overline{K}(t) =: K(-t)$, and $f\perp B_E^2$ means $\int_{\mathbb{R}} f(t)\varphi(t)dt=0$ for any $\varphi\in B_E^2$.

Thus, Lemma 3.1 follows from (3.5).

Proof of Theorem 2 The last inequlity of (1.5) easily follows from Lemma 3.1 if we set $E=[-\pi\sigma,\pi\sigma]$. Next, we prove the first inequlity of (1.5). Let A be a subspace of average dimension $\le\sigma$. Since $\overline{\dim}(B_{\pi c}^2,L^2)=c$, then it is easy verify that $A+B_{\pi c}^2$ is a subspace of average dimension $\le \sigma+c$ of L^2, where $A+B_{c\pi}^2=\{f+g : f\in A, g\in B_{c\pi}^2\}$.

Hence, we have

$$\sqrt{2\pi}E(W_1^rF,A,L^2)\ge\sqrt{2\pi}E(W_1^rF,A+B_{c\pi}^2,L^2)\ge E(B_1^0(K),A+B_{c\pi}^2,L^2). \tag{3.6}$$

Set

$$g_y(x)=\frac{K(x+y/2)-K(x-y/2)}{2}.$$

When $r>1$, it is easy to verify that under the conditions of Theorem 2, K instead of G in Theorem 1 satisfies the conditions of Theorem 1. By [14],[16], similar to the proof of Theorem 1, we have

$$E(B_1^0(K),A+B_{c\pi}^2,L^2)\ge\sup_y e(g_y(\cdot),A+B_{c\pi}^2,L^2) \tag{3.7}$$

and

$$\sup_y(1+\epsilon)e(g_y(\cdot),A+B_{c\pi}^2,L^2)+\epsilon\|K\|_2\ge\sup_{y\in\mathbb{R}}\inf_{\text{mes } E\le 2\pi(\sigma+c)}\|(g_y)^\wedge(\cdot)\|_{L^2(\mathbb{R}-E)} \tag{3.8}$$

for any $\epsilon>0$.

By (3.6) and (3.8), we have

$$\sqrt{2\pi}E(W_1^rF,A,L^2)\ge\sup_{y\in\mathbb{R}}\inf_{\text{mes } E\le 2\pi(\sigma+c)}\|(g_y)^\wedge(\cdot)\|_{L^2(\mathbb{R}-E)}. \tag{3.9}$$

When $r=1$, set

$$\chi_N(t)=\begin{cases}\eta_c(t+N), & \text{if } -N\le t\le 0,\\ \eta_c(N-t), & \text{if } 0\le t\le N,\\ 0, & \text{if } |t|>N.\end{cases} \tag{3.10}$$

Obviously, $0 \leq \chi_N(t) \leq 1, \sup p(\chi_N) \subset [-N, N], \chi_N(t) = 1, t \in [-N + c\pi, N - c\pi]$, and $\chi_N(t)$ has continuous and bounded second derivative.

Denote by

$$K_N(x) =: \frac{1}{\sqrt{2\pi}} \int_{I\!R} \frac{\eta_c(t) F(t) \chi_N(t)}{it} e^{itx} dt, \tag{3.11}$$

and

$$(g_y)_N(x) =: \frac{K_N(x + y/2) - K_N(x - y/2)}{2}. \tag{3.12}$$

For any N, it is easy to see that $K_N(x)$ instead of C in Theorem 1 satisfies the conditions of Theorem 1, we have

$$\begin{aligned} \sqrt{2\pi} E(W_1^1 F, A, L^2) &\geq E(B_1^0(K), A + B_{c\pi}^2, L^2) \\ &\geq E((B_1^0)(K_N), A + B_{c\pi}^2, L^2) - \|K - K_N\|_2 \\ &\geq \sup_{y \in I\!R} \inf_{\text{mes } E \leq 2\pi(\sigma + c)} \|(g_y)_N^\wedge\|_{L^2(I\!R - E)} - \|K - K_N\|_2. \end{aligned} \tag{3.13}$$

Notice that $\|K - K_N\|_2 \to 0$ as $N \to \infty$. Letting $N \to \infty$ in (3.13), we also get (3.9) for $r = 1$.

Since $|F(x)|$ is even on $I\!R$ and nonincreasing on $(0, \infty)$, then by a similar argument as used by Pinkus in [10], we may obtain

$$\frac{|F(t)|^2}{t^{2r}}(1 - \cos ty) \leq \frac{|F(t)|^2}{t^{2r-2}} \min_{|t| \leq \pi(\sigma + 2c)} \frac{1 - \cos ty}{t^2}, \tag{3.14}$$

for $|t| \geq \pi(\sigma + 2c)$ and $|y| \leq \frac{1}{\sigma + 2c}$.

Hence, by (3.14), we have

$$\sup_{y \in I\!R} \inf_{\text{mes } E \leq 2\pi(\sigma + c)} \|g_y^\wedge(\cdot)\|_{L^2(I\!R - E)} \geq \sup_{|y| \leq \frac{1}{\sigma + 2c}} \{2 \int_{\pi(\sigma + 2c)}^\infty \frac{|F(t)|^2}{t^{2r}} (1 - \cos ty) dt\}^{1/2}, \tag{3.15}$$

By (3.9) and (3.15), we have

$$\sqrt{2\pi} E(W_1^r F, A, L^2) \geq \sup_{|y| \leq \frac{1}{\sigma + 2c}} \{2 \int_{\pi(\sigma + 2c)}^\infty \frac{|F(t)|^2}{t^{2r}} (1 - \cos ty) dt\}^{1/2}, \tag{3.16}$$

Letting $c \to 0^+$ in (3.16), the first inequality of (1.5) is obtained. We complete the proof of Theorem 2.

4. Some Corollaries

Let $P_r(t) = \prod_{j=1}^r (t - t_j), t_j \in I\!R$, be a polynomial of degree r with only real zeros, and $P_r(D), D = \frac{d}{dx}$, the differential operator corresponding $P_r(t)$. Set

$$W_1^r(P_r) = \{f \in L^2; \|P_r(D) f\|_1 \leq 1\}.$$

Obviously, $P_r(iy) f^\wedge(y) = (P(D)f)^\wedge(y), y \in I\!R$, for any $f \in W_1(P_r)$, we have

Theorem 3 *Let $r \in Z_+$. Then,*

(i) when $P_r(0) = 0$,

$$\sup_{|y|<\frac{1}{\sigma}} \{\frac{1}{\pi}\int_{\pi\sigma}^{\infty} \frac{1-\cos ty}{|P^r(it)|^2} dt\}^{1/2} \le \bar{d}_\sigma(W_1(P_r), L^2)$$
$$\le E(W_1(P_r), B^2_{\pi\sigma}, L^2) \le \sup_{y\in I\!R}\{\frac{1}{\pi}\int_{\pi\sigma}^{\infty} \frac{(1-\cos ty)dt}{|P_r(t)|^2}\}^{1/2}.$$

(ii) when $P_r(0) \neq 0$,

$$\bar{d}_\sigma(W_1(P_r), L^2) = E(W_1(P_r), B^2_{\sigma\pi}, L^2) = \{\frac{1}{\pi}\int_{\pi\sigma}^{\infty} \frac{dt}{|P_r(it)|^2}\}^{1/2}.$$

Corollary *Let $r \in Z_+$. If $P_r(t) = t^r$, then*

$$\sup_{|y|\le\frac{1}{2}} \{\frac{1}{\pi}\int_{\pi\sigma}^{\infty} \frac{1-\cos ty}{t^{2r}} dt\}^{1/2} \le \bar{d}_\sigma(W_1^r(I\!R), L^2)$$
$$\le E(W_1^r(I\!R), B^2_{\pi\sigma}, L^2) \le \sup_{y\in I\!R}\{\frac{1}{\pi}\int_{\pi\sigma}^{\infty} \frac{(1-\cos ty)dt}{t^{2r}}\}^{1/2}.$$

Let G be a PF density (Shortly, $G \in PF$) and $P(z)$ the reciprocal of the Laplace transformation of $G(x)$. It is well-known that $P(z)$ may be represented as

$$P(z) = e^{-cz^2+bz}\prod_{k=1}^{\infty}(1-\frac{z}{a_k})\cdot e^{-\frac{z}{a_k}}, \tag{4.1}$$

where $c \ge 0, b, a_k \in I\!R, 0 < c + \sum_{k=1}^{\infty} a_k^{-2} < \infty$. Moreover,

$$G(x) = \frac{1}{2\pi}\int_{I\!R} \frac{e^{iux}}{P(iu)} du. \tag{4.2}$$

By [7],[12], we see

$$|G(x)| = O(e^{-c|x|}), |x| \to \infty.$$

for some $c > 0$. Meanwhile, it is easy to see that $\frac{1}{|P(iu)|}$ is even on $I\!R$ and nonincreasing on $(0,\infty)$. Thus, we have

Theorem 4 *Let G and P be defined by (4.1) and (4.2). Then*

$$\bar{d}_\sigma(B_1(G), L^2) = E(B_1(G), B^2_{\pi\sigma}, L^2) = \{\frac{1}{\pi}\int_{\pi\sigma}^{\infty} \frac{du}{|P(iu)|^2}\}^{1/2}.$$

Let $P(z)$ be defined by (4.1). Set $W_1^r(P) = \{f \in L^2$:there exists $h \in L^1$ such that $(it)^r P(it) f^\wedge(t) = h^\wedge(t)$ *a.e.* $t \in I\!R\}$.

By Theorem 2, we have

Theorem 5 *If $P(z)$ is defined by (4.1), then*

$$\sup_{|y|\le\frac{1}{2}}\{\frac{1}{\pi}\int_{\pi\sigma}^{\infty}\frac{1-\cos ty}{t^{2r}|P(it)|^2}dt\}^{1/2}\le\bar{d}_\sigma(W_1^r(P),L^2)$$
$$\le E(W_1^r(P),B^2_{\pi\sigma},L^2)\le\sup_{y\in\mathbb{R}}\{\frac{1}{\pi}\int_{\pi\sigma}^{\infty}\frac{(1-\cos ty)dt}{t^{2r}|P(it)|^2}\}^{1/2}.$$

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$L(\mathbb{R})$ 中卷积函数类在 $L_2(\mathbb{R})$ 中的平均 $\sigma-K$ 宽度

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摘　要

本文研究了 $L(\mathbb{R})$ 中的卷积函数类在 $L_2(\mathbb{R})$ 中的平均 $\sigma-K$ 宽度且得到一些精确结果. 同时，对非卷积情形，得到一些渐近结果.