Proof Let $f \in B_{p,w}$. Using Lemma 2.1, we have

$$Tf(x) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} d_{kj}a_{kj}(x),$$

where a_{kj} is a $H_w^{1,p}$-atom, and $d_{kj} = m_k\mu_j(b_k), |\mu_j(b_k)| \le C\theta(\frac{1}{2^j}), k \in N, j \in N$. Thus

$$\begin{aligned} M(\{d_{kj}\}) &\le C\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}|m_k|\theta(\frac{1}{2^j})(1+\log^+(\frac{1}{C|m_k|\theta(\frac{1}{2^j})})) \\ &\le C\sum_{k=1}^{\infty}|m_k|(1+\log^+(\frac{1}{|m_k|}))\sum_{j=1}^{\infty}\theta(\frac{1}{2^j})(1+\log^+(\frac{1}{C\theta(\frac{1}{2^j})})), \end{aligned}$$

where

$$\sum_{j=1}^{\infty}\theta(\frac{1}{2^j})(1+\log^+(\frac{1}{C\theta(\frac{1}{2^j})})) \le C\int_0^1(\frac{\theta(t)}{t}\log^+\frac{1}{\theta(t)})dt < \infty.$$

Thus $Tf(x) \in \mathring{B}_{p,w}$.

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关于广义Calderón-Zygmund算子

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摘 要

本文讨论了$\theta(t)$ 型和$(\log,\theta)$ 型Calderón-Zygmund 算子在加权Hardy 型空间HA_w^p 上的有界性，$\theta(t)$ 型Calderón-Zygmund 算子在Hardy 型加权块空间$\mathring{B}_{p,w}$ 上的有界性，以及广义的w-Calderón-Zygmund 算子是$H^A p_w$ 到HA^p 上的有界算子.

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On Generallzed Calderón-Zygmund Operators *

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Abstract In this paper, we discussed the boundedness of Calderón-Zygmund operators of type $\theta(t)$ and $(\log,\theta)$ on HA_w^p and the boundedness of Calderón-Zygmund operators of type $\theta(t)$ on $B_{p,w}$ and the boundedness of the generalized w-Calderón-Zygmund operators from HA_w^p to HA^p.

Keywords Calderón-Zygmund operators, HA_w^p spaces, $B_{p,w}$ spaces, HA^p spaces.

Classification AMS(1991) 42B20/CCL O177.6

1. Introduction

The Claderón-Zygmund operator of type $\theta(t)$ and $(\log,\theta)$, and the generalized w-Calderón-Zygmund operators are as following[1][2][3]:

If T is a bounded operator from the class $\phi(R^n)$ of Schwarts functions to its dual $\phi'(R^n)$, satisfying the following conditions:

(1.1) There exists $C>0$, such that for any $f \in C_0^\infty(R^n)$

$$\|Tf\|_{L^2(R^n)} \le C\|f\|_{L^2(R^n)}.$$

(1.2) There exists a continuous function $K(x,y)$ defined on $\Omega = R^n \times R^n \backslash \{x=y\}$ and $C>0$ such that

(1.a) $|K(x,y)| \le C|x-y|^{-n}$ for all $(x,y) \in \Omega$;

(1.b) for all x, x_0, y with $2|x-x_0| < |x_0-y|$

$$|K(x,y) - K(x_0,y)| + |K(y,x) - K(y,x_0)| \le C\theta\Big(\frac{|x_0-x|}{|x_0-y|}\Big)|x_0-y|^{-n},$$

where $\theta(t)$ is a nonnegative nondecreasing function on $[0,+\infty)$ with $\int_0^1 \frac{\theta(t)}{t}dt < \infty$ and $\theta(0)=0, \theta(2t) \le C\theta(t)$;

(1.c) $Tf(x) = \int K(x,y)f(y)dy$ a.e. $x \notin \operatorname{supp} f$.

Then T is said to be a Calderón-Zygmund operator of type $\theta(t)$.

The Calderón-Zygmund operator of type $(\log,\theta)$ is as above and (1.d) insdead of (1.b):

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(1.d) for all $x, x_0.y$, with $2|x-x_0| < |y-x_0|$

$$|K(x,y)-K(x_0,y)|+|K(y,x)-K(y,x_0)| \le C[\theta(\frac{|x_0-x|}{|x_0-y|})|x_0-y|^{-n}+G(x,x_0,y)],$$

where $G(x,x_0,y)$ is a function satisfying

$$G(x,x_0,y) \le \begin{cases} |x_0-y|^{-n}(\log \dfrac{2}{|x_0-x|})^{-1} & \text{if } |x_0-x| \le 1 \text{ and } |x_0-y| \le 2\sqrt{n}, \\ |x_0-y|^{-n}\theta(\dfrac{\max\{|x_0-x|,1\}}{|x_0-y|}) & \text{if } |x_0-y| \ge 1. \end{cases}$$

The generalized w-Calderón-Zygmund operator is (1.e) and (1.f) insdead of (1.a) and (1.b):

(1.e) $|K(x,y)| \le Cw_{|x-y|}(x)|x-y|^{-n}$ for all $(x,y) \in \Omega$;

(1.f) for all x, x_0, y with $2|x-x_0| < |x_0-y|$

$$|K(x,y)-K(x_0,y)|+|K(y,x)-K(y,x_0)| \le Cw_{|x-x_0|}(x)\theta(\frac{|x_0-x|}{|x_0-y|})|x_0-y|^{-n},$$

where $w \in A_1$ (Muckenhoupt class), and $w_t(x) = t^{-n}\int_{|x-y|<t} w(y)dy$.

HA^p_w spaces, $\overset{\circ}{B}_{p,w}$ spaces and HA^p spaces are as following:

(1.3) $HA^p_w(R^n) = \{f \in L^1(R^n) : f(x) = \sum_k \lambda_k a_k(x), \sum_k |\lambda_k| < \infty\}$ where $a_k(x)$ is a weighted central $(1,p)$-atom, $k = 1,2,3,\cdots, 1 < p < \infty, \|f\|_{HA^p_w} = \inf(\sum_k |\lambda_k|)$;

(1.4) $\overset{\circ}{B}_{p,w}(R^n) = \{f \in L^1_w(R^n) : f(x) = \sum_k m_k b_k(x), M(\{m_k\}) < \infty\}$ where $b_k(x)$ is a $H^{1,p}_w$- atom, $M(\{m_k\}) = \sum_k |m_k|(1+\log^+(\frac{1}{|m_k|}))$;

(1.5) $HA^p(R^n) = \{f \in L^1(R^n) : f(x) = \sum_k \mu_k h_k(x), \sum_k |\mu_k| < \infty\}$ where $h_k(x)$ is a $(1,p)$- atom, $k = 1,2,3,\cdots$.

2. Lemmas

Lemma 2.1 *Let $b(x)$ be a weighted central $(1,p)$-atom (or $H^{1,p}_w$-atom), and T be a Calderon-Zygmund operator of type $\theta(t)$, and $T^*(1) = 0$. Then*

(2.2) $Tb(x) = \sum_{k \in N} \mu_k(b) a_k(x,b)$ and $|\mu_k(b)| \le C\theta(\frac{1}{2^k})$,

where $a_k(x,b)$ is the same type atom as $b(x)$.

Proof Let $b(x)$ be as following:

(2.a) $\text{supp} b \subset B(x_0,r)$ (B is a ball);

(2.b) $\|b\|_{p,w} \le w(B(x_0,r))^{\frac{1}{p}-1}$;

(2.c) $\int b(x)dx = 0$.

Write $E_k = \{x \in R^n : 2^{k-1}r < |x_0-x| \le 2^k r\}, k = 1,2,3,\cdots, B_k = \{x \in R^n : |x_0-x| \le$

$2^k r\}, k=0,1,2,\cdots$, and $Tb(x)=\chi_{B_1}(x)Tb(x)+\sum_{k=2}^{\infty}\chi_{E_k}(x)Tb(x)\triangleq\sum_{k=1}^{\infty}b_k(x)$. Then

$$\|b_1\|_{p,w}\le\|Tb\|_{p,w}\le C\|b\|_{p,w}\le Cw(B)^{\frac{1}{p}-1}\le C(\frac{w(B_1)}{w(B)})^{1-\frac{1}{p}}w(B_1)^{\frac{1}{p}-1}.$$

Now write $\mu_1(x_0,r)=C(\frac{w(B_1)}{w(B)})^{1-\frac{1}{p}}, a_1(x)=\frac{b_1(x)}{\mu_1(x_0,r)}$, thus supp $a_1\subset B_1, \|a_1\|_{p,w}\le w(B_1)^{\frac{1}{p}-1}$, for $k>1$

$$\begin{aligned}\|b_k\|_{p,w} &\le (\int_{E_k}|Tb(x)|^p w(x)dx)^{\frac{1}{p}}\\ &\le (\int_{B_k}(\int_B C\theta(\frac{|x_0-t|}{|x_0-x|})|x_0-x|^{-n}|b(t)|dt)^p w(x)dx)^{\frac{1}{p}}\\ &\le C\theta(\frac{1}{2^k})(2^k r)^{-n}(\int_{E_k}(\int_B|b(t)|dt)^p w(x)dx)^{\frac{1}{p}}\\ &\le C\theta(\frac{1}{2^k})\frac{|B|}{|B_k|}(\frac{w(B_k)^{\frac{1}{p}}}{w(B)})\le C\theta(\frac{1}{2^k})w(B_k)^{\frac{1}{p}-1},\end{aligned}$$

if we write $\mu_k(x_0,r)=C\theta(\frac{1}{2^k}), a_k(x)=\frac{b_k(x)}{\mu_k(x_0,r)}, k=2,3,4,\cdots$, then supp$a_k\subset B_k$, and $\|a_k\|_{p,w}\le w(B_k)^{\frac{1}{p}-1}, k=2,3,4,\cdots$. Thus $Tb(x)=\sum_{k=1}^{\infty}\mu_k(x_0,r)a_k(x)$.

Now, let $F_1=B_1, F_k=E_k, k>1$, and

$$\begin{aligned}Tb(x) &= \sum_{k=1}^{\infty}(b_k(x)-\frac{\chi_{F_k}(x)}{|F_k|}\int b_k(t)dt\}+\sum_{k=1}^{\infty}\frac{\chi_{F_k}(x)}{|F_k|}\int b_k(t)dt\\ &= \sum_{k=1}^{\infty}(b_k(x)-\frac{\chi_{F_k}(x)}{|F_k|}\int b_k(t)dt+\sum_{k=1}^{\infty}(\sum_{j=k+1}^{\infty}\int_{F_j}b_j)(\frac{\chi_{F_{k+1}}(x)}{|F_{k+1}|}-\frac{\chi_{F_k}(x)}{|F_k|})\\ &\quad+(\sum_{j=1}^{\infty}\int_{F_j}b_j)\frac{\chi_{F_k}(x)}{|F_k|}\triangleq \mathrm{I}+\mathrm{II}+\mathrm{III}\end{aligned}$$

and

$$I=\sum_{k=1}^{\infty}\mu_k(x_0,r)\{a_k(x)-\frac{\chi_{F_k}(x)}{|F_k|}\int a_k\}\triangleq 2\sum_{k=1}^{\infty}\mu_k(x_0,r)\bar{a}_k(x),$$

we know that every $\bar{a}_k(x)$ is an atom, and $|\mu_1|\le C, |\mu_k|\le C\theta(\frac{1}{2^k}), k=2,3,4,\cdots$,

$$\mathrm{II}\le\sum_{k=1}^{\infty}(\sum_{j=k+1}^{\infty}\int_{F_j}b_j)\frac{w(B_{k+1})}{|B_{k+1}|}\bar{\bar{a}}_k(x),$$

we know that every $\bar{\bar{a}}(x)$ is an atom too, and

$$\sum_{j=k+1}^{\infty}|\int_{F_j}b_j|\frac{w(B_{k+1})}{|B_{k+1}|}\le C\frac{w(B_{k+1})}{|B_{k+1}|}\sum_{j=k+1}^{\infty}(\int_{B_j}|b_j(t)|^p dt)^{\frac{1}{p}}|B_j|^{\frac{1}{p'}}$$

$$\le C\frac{w(B_{k+1})}{|B_{k+1}|}\sum_{j=k+1}^{\infty}(\frac{1}{\text{essin } fw})^{\frac{1}{p}}\|b_j\|_{p,w}|B_j|^{\frac{1}{p'}}$$

$$\le C\frac{w(B_{k+1})}{|B_{k+1}|}\sum_{j=k+1}^{\infty}(\frac{|B_j|}{w(B_j)})^{\frac{1}{p}}\theta(\frac{1}{2^j})\frac{|B|w(B_j)^{\frac{1}{p}}}{|B_j|w(B)}|B_j|^{\frac{1}{p'}}\le C\theta(\frac{1}{2^k}).$$

For III $\sum_{k=1}^{\infty}\int_{F_j} b_j=\int Tb(x)dx=(Tb,1)=(b,T^*1)=0$, thus III satisfies (2.2).

Lemma 2.3 *Let $a(t)$ be a weighted central $(1,p)$-atom, and T be a Calderón-Zygmund operators of type $(\log,\theta)$, or T be a generalized w-Calderón-Zygmund operator, then*

(2.4) $$Ta(x)=\sum_{k\in N}\lambda_k(a)b_k(x,a) \text{ and } |\lambda_k(a)|\le C\theta(\frac{1}{2^k})\quad (T^*(1)=0),$$

where $b_k(x,a)$ is a $(1,p)$-atom, $k\in N$.

The proof is similar to the proof of Lemma 2.1.

3. Theorems

Theorem 3.1 *Let $w\in A_1$ (Muchenhoupt class) and T be a Calderón-Zygmund operators of type $\theta(t), T^*(1)=0$, then T is bounded on $HA_w^p(R^n)$.*

Proof Let $f\in HA_w^p(R^n)$. Using Lemma 2.1, we have

$$Tf(x)=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}c_{kj}a_{kj}(x),$$

where a_{kj} is a $(1,p)$-atom, and $c_{kj}=\lambda_k\mu_j(b_k), |\mu_j(b_k)|\le C\theta(\frac{1}{2^j}), j\in N$. Thus

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}|c_{kj}|\le C(\sum_{k=1}^{\infty}|\lambda_k|)(\sum_{j=1}^{\infty}\theta(\frac{1}{2^j})),$$

where $\sum_{j=1}^{\infty}\theta(\frac{1}{2^j})\le C\int_0^1\frac{\theta(t)}{t}dt<\infty$, thus $Tf(x)\in HA_w^p$.

Theorem 3.2 *Let T be a Calderón-Zygmund operators of type $(\log,\theta)$, and $T^*(1)=0$, then T is bounded on $HA_w^p(R^n)$.*

Theorem 3.3 *Let T be a generalized w-Calderón-Zygmund operator, and $T^*(1)=0$, then T is bounded from HA_w^p to HA^p.*

The proof of Theorem 3.2 and Theorem 3.3 is same as above and Lemma 2.3 insdead of Lemma 2.1, and using [3].

Theorem 3.4 *Suppose that T and $\theta(t)$ are as Theorem 3.1 and $\theta(t)$*

$$\int_0^1(\frac{\theta(t)}{t}\log^+\frac{1}{\theta(t)})dt<\infty$$

Then T is bounded on $\mathring{B}_{p,w}$.

Proof Let $f \in B_{p,w}$. Using Lemma 2.1, we have

$$Tf(x) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} d_{kj} a_{kj}(x),$$

where a_{kj} is a $H_w^{1,p}$-atom, and $d_{kj} = m_k \mu_j(b_k), |\mu_j(b_k)| \leq C\theta(\frac{1}{2^j}), k \in N, j \in N$. Thus

$$\begin{aligned} M(\{d_{kj}\}) &\leq C\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |m_k|\theta(\frac{1}{2^j})(1+\log^+(\frac{1}{C|m_k|\theta(\frac{1}{2^j})})) \\ &\leq C\sum_{k=1}^{\infty} |m_k|(1+\log^+(\frac{1}{|m_k|}))\sum_{j=1}^{\infty}\theta(\frac{1}{2^j})(1+\log^+(\frac{1}{C\theta(\frac{1}{2^j})})), \end{aligned}$$

where

$$\sum_{j=1}^{\infty}\theta(\frac{1}{2^j})(1+\log^+(\frac{1}{C\theta(\frac{1}{2^j})})) \leq C\int_0^1(\frac{\theta(t)}{t}\log^+\frac{1}{\theta(t)})dt < \infty.$$

Thus $Tf(x) \in \mathring{B}_{p,w}$.

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关于广义 Calderón-Zygmund 算子

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摘　要

本文讨论了$\theta(t)$ 型和$(\log,\theta)$ 型Calderón-Zygmund 算子在加权Hardy 型空间HA_w^p 上的有界性，$\theta(t)$ 型Calderón-Zygmund 算子在Hardy 型加权块空间$\mathring{B}_{p,w}$ 上的有界性，以及广义的w-Calderón-Zygmund 算子是$H^A p_w$ 到HA^p 上的有界算子.