Hence if $|x-c|<1$, $\lim_{n\to\infty}(n+2)^{1+\frac{1}{n+1}}|x-c|^{n+1}=0$, which implies that f is analytic on I. For each c, the radius of convergence r_c of the power series expansion about c is $\geq 1+|c|$, again by Lemma 1. Thus f is analytic in $\cup\{N_c : c\in I\}$, where $N_c=\{z : |z-c|<1+|c|\}$. But it is easily seen that $\cup N_c=E$, and this completes the proof.

References

[1] A.L. Horwitz and L.A. Rubel, *Two theorems on inverse interpolation*, Rocky Mountain Math., **18**(3)(1988),645–653.

[2] A.L. Horwitz and L.A. Rubel, *Totally positive functions and totally bounded functions on* $[-1,1]$, J. of Approx. theory 52(1988) 204–216.

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完全BMO-有界函数

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摘　要

定义于区间$I=[-1,1]$上的实值函数f,若它的一切Lagrange插值多项式在BMO(I)范数下一致有界，则称f为完全BMO-有界函数. 本文引入这一概念并讨论这类函数的性质.

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Totally BMO-Bounded Functions on $[-1,1]$ *

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Abstract A real-valued function f defined on $I=[-1,1]$ is said to be totally BMO-bounded if there exists a positive constant M such that $\|p\|_{\mathrm{BMO}(I)} \leq M$ for each Lagrange interpolant p of f. This class of functions is studied here.

Keywords Lagrange interpolant, inverse interpolation, totally BMO-bounded functions
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1.Introduction

This paper focuses on inverse interpolation begun in [1] and developed in [2]. What we mean by inverse interpolation is to deduce some property of a function f from property or some properties of its set $\mathcal{L}(f)$ of Lagrange interpolants. Before we give more precise definitions,we need some preliminaries.

Let n be a nonnegative integer. If f is a real-valued function on $I=[-1,1]$, we say that a polynomial p of degree n is a Lagrange interpolant of f if there are $n+1$ distinct numbers $\{x_0,x_1,\cdots,x_n\}\subseteq I$ such that $p(x_j)=f(x_j)$ for $j=0,1,\cdots,n$. We often use the Newton form for the interpolating polynomial:

$$p(x)=f(x_0)+f[x_1,x_0](x-x_0)+\cdots+f[x_n,\cdots,x_1,x_0](x-x_0)\cdots(x-x_{n-1}).$$

We use the notation $p(x)=L(f;x_0,\cdots,x_n)$, where $f[x_j,\cdots,x_0]$ is just the well-known jth-order divided-difference of f which is defined inductively by

$$f[x_j,\cdots,x_0]\equiv\frac{f[x_j,\cdots,x_1]-f[x_{j-1},\cdots,x_0]}{x_j-x_0}.$$

In addition, the set of all Lagrange interpolants of f is denoted by $\mathcal{L}(f)$.

Definition 1(see [2]) *A real-valued function f defined on I is said to be totally bounded*

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if there exists a constant M such that $|p(x)| \le M$ for all $p \in \mathcal{L}(f)$ and all $x \in I$. We write

$$\| f \|_{\text{TBI}} = \sup_{\substack{x\in I \\ p\in\mathcal{L}(f)}} |p(x)|$$

and denote the class of all such functions by TBI.

TBI is a Banach space with the norm $\|\cdot\|_{\text{TBI}}$ and the condition of Definition 1 implies that f is analytic in a certain region E containing I (see [2]).

Definition 2(see [4]) BMO*(I) denotes bounded mean oscillation space defined on I. $f \in$BMO(I) if and only if*

$$\|f\|_{\text{BMO(I)}} = \sup_{[c,d]\subset I} \frac{1}{d-c}\int_c^d \left| f(y) - \frac{1}{d-c}\int_c^d f(x)\,\mathrm{d}x \right| \mathrm{d}y < \infty. \tag{1.1}$$

Clearly,

$$\begin{aligned} \|f\|_{\text{BMO(I)}} &= 0 \Longleftrightarrow f = \text{constant}, \\ \|f\|_{\text{BMO(I)}} &\le 2\|f\|_{\infty,I}, \end{aligned}$$

where

$$\|f\|_{\infty,I} = \sup_{x\in I} |f(x)|. \tag{1.2}$$

In fact, Definition 1 implies that $\mathcal{L}(f)$ is a bounded set in $C(I)$. At this point it is natural to ask: What can be said about f if $\mathcal{L}(f)$ is a bounded set in BMO(I) ? We intrduce the following concept.

Definition 3 *A real-valued function f defined on I is said to be totally* BMO*-bounded if there exists a constant M such that $\|p\| \le M$ for all $p \in \mathcal{L}(f)$. We write*

$$\|f\|_{\text{TB}_{\text{BMO}}\text{I}} = \sup_{p\in\mathcal{L}(f)} \|p\|_{\text{BMO(I)}}$$

and denote the class of all such functions by $\text{TB}_{\text{BMO}}I$.

Obviously, $\|\cdot\|_{\text{TB}_{\text{BMO}}}I$ gives a norm and $\text{TB}_{\text{BMO}}I$ is a normed linear space. We have from (1.2) that TBI $\subset \text{TB}_{\text{BMO}}$I. This paper will focus on TB_{BMO}I.

2.Totally BMO-bounded functions on I.

Theorem 1 *If $f \in \text{TB}_{\text{BMO}}I$, Then $f \in C^\infty(I)$.*

Proof By [3] and the proof of Theorem 2 in [2], it suffices to prove that for any given positive integer n,there exists a positive constant M_n only depending on n such that

$$|f[x_0,\cdots,x_n]| \le M_n \quad \text{for all choices of points} \quad -1 \le x_0 < x_1 < \cdots < x_n \le 1. \tag{2.1}$$

First we have

$$f \text{ is bounded on } I. \tag{2.2}$$

To prove (2.2), suppose that $f(x^{(j)}) \longrightarrow \infty$, and $f(0)=0$ (otherwise, considering $f(x)-f(0)$). Consider $p_j(x)=\frac{f(x^{(j)})}{x^{(j)}}x$, the linear interpolant to f at $\{0, x^{(j)}\}$.

Since $\|x\|_{\mathrm{BMO(I)}}>0$, we have $\|p_j\|_{\mathrm{BMO(I)}}=\left|\frac{f(x^{(j)})}{x^{(j)}}\right|\|x\|_{\mathrm{BMO(I)}} \longrightarrow \infty$,a contradiction. Now we make the following inductive hypothesis:

$$|f[x_0,\cdots,x_n]| \le M_n$$

for all choices of points $-1 \le x_0 < x_1 < \cdots < x_n \le 1$.

Suppose that $|f[x_0^{(j)},\cdots,x_{n+1}^{(j)}]| \longrightarrow \infty$ for some sequence

$$\{x^{(j)}\}, x^{(j)}=(x_0^{(j)},\cdots,x_{n+1}^{(j)}) \in I^{n+2},$$

with all coordinates distinct. Taking subsequences if necessary, assume $\{x_{(j)}\} \longrightarrow x=(x_0,\cdots,x_{n+1})$.Consider

$$\begin{aligned} p_j(x) &\equiv L(f;x_0^{(j)},\cdots,x_{n+1}^{(j)}) \\ &= f(x_0^{(j)})+\cdots+f[x_0^{(j)},\cdots,x_n^{(j)}](x-x_0^{(j)})\cdots(x-x_{n-1}^{(j)}) \\ &\quad + f[x_0^{(j)},\cdots,x_n^{(j)},x_{n+1}^{(j)}](x-x_0^{(j)})\cdots(x-x_{n-1}^{(j)})(x-x_n^{(j)}). \end{aligned}$$

We have from (1.2) that

$$\begin{aligned} \|p_j\|_{\mathrm{BMO(I)}} \ge\ & |f[x_0^{(j)},\cdots,x_{n+1}^{(j)}]|\,\|(x-x_0^{(j)})\cdots(x-x_n^{(j)})\|_{\mathrm{BMO(I)}} \\ & -(2^2M_1+\cdots+2^{n+1}M_n), \end{aligned}$$

and

$$\begin{aligned} &\lim_{j\to\infty}\|(x-x_0^{(j)})(x-x_1^{(j)})\cdots(x-x_n^{(j)})\|_{\mathrm{BMO(I)}} \\ &= \|(x-x_0)(x-x_1)\cdots(x-x_n)\|_{\mathrm{BMO(I)}} > 0. \end{aligned}$$

Then $\|p_j\|_{\mathrm{BMO(I)}} \longrightarrow \infty$, which is a contradition. Hence we have that $|f[x_0,\cdots,x_{n+1}]| \le M_{n+1}$ for all points x_j such that $-1 \le x_0 < \cdots < x_{n+1} \le 1$. So by induction (using (2.2) to get started), for each positive integer n, $|f[x_0,\cdots,x_n]| \le M_n$.

Lemma 1 *For any* $f \in \mathrm{TB_{BMO}I}$,

$$\left|\frac{f^{(n)}(x)}{n!}\right| \le \frac{4\|f\|_{\mathrm{TB_{BMO}I}}(n+1)^{1+\frac{1}{n}}}{(1+|x|)^n}.$$

Proof Consider for any $x \in I$ the Taylor interpolant

$$s_n(x;c)=f(c)+f'(c)(x-c)+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n$$

By taking limits, if $f \in \mathrm{TB_{BMO}I}$,then

$$\|s_n(x;c)\|_{\mathrm{BMO(I)}} \le \|f\|_{\mathrm{TB_{BMO}I}}\cdot$$

Hence

$$\|s_n(x;c) - s_{n-1}(x;c)\|_{\mathrm{BMO(I)}} \le 2\|f\|_{\mathrm{TB_{BMO}I}}.$$

So that

$$\|\frac{f^{(n)}(c)}{n!}(x-c)^n\|_{\mathrm{BMO(I)}} \le 2\|f\|_{\mathrm{TB_{BMO}I}}.$$

Moreover

$$\left|\frac{f^{(n)}(c)}{n!}\right| \le \frac{2\|f\|_{\mathrm{TB_{BMO}I}}}{\|(x-c)^n\|_{\mathrm{BMO(I)}}}. \tag{2.3}$$

It suffices to estimate $\|(x-c)^n\|_{\mathrm{BMO(I)}}$. If $c \ge 0$, then

$$\frac{1}{c+1}\int_{-1}^{c}(x-c)^n\,\mathrm{d}x = \frac{(-1)^n}{n+1}(1+c)^n,$$

$$\frac{1}{c+1}\int_{-1}^{c}\left|(x-c)^n - \frac{(-1)^n}{n+1}(1+c)^n\right| dx = \frac{1}{c+1}\int_{-1}^{c}\left|(c-x)^n - \frac{1}{n+1}(c+1)^n\right| dx.$$

Obviously, there exists a root of $(c-x)^n - \frac{1}{n+1}(c+1)^n$ in $(-1,c)$ which is denoted by c^*. Thus

$$\begin{aligned}\|(x-c)^n\|_{\mathrm{BMO(I)}} &\ge \frac{1}{c+1}\int_{-1}^{c}\left|(c-x)^n - \frac{1}{n+1}(c+1)^n\right| \mathrm{d}x\\ &\ge \frac{1}{c+1}\int_{c^*}^{c}\left[\frac{1}{n+1}(c+1)^n - (c-x)^n\right]\mathrm{d}x\\ &= \left[\frac{1}{(n+1)^{1+\frac{1}{n}}} - \frac{1}{(n+1)^{2+\frac{1}{n}}}\right](c+1)^n\\ &\ge \frac{(1+|c|)^n}{2(n+1)^{1+\frac{1}{n}}},\end{aligned} \tag{2.4}$$

Similarly, (2.4) is true when $c < 0$.Then we complete the proof from (2.3) and (2.4).

Theorem 2 *Let E denote union of the two discs in the complex plane $E_1 = \{z :|z-1| < 2\}$ and $E_2 = \{z :|z+1| < 2\}$. Then if $f \in \mathrm{TB_{BMO}}I$, f may be extend to be analytic in E.*

Proof For any $c \in I$,

$$f(x) - s_n(x;c) = \frac{1}{n!}\int_c^x f^{n+1}(t)(x-t)^n\,\mathrm{d}t.$$

By Lemma 1, we have

$$\begin{aligned}|f(x) - s_n(x;c)| &\le \frac{1}{n!}4\|f\|_{\mathrm{TB_{BMO}I}}(n+1)!\int_c^x \frac{(n+2)^{1+\frac{1}{n+1}}|(x-t)^n|}{(1+|t|)^{n+1}}\mathrm{d}t\\ &\le 4\|f\|_{\mathrm{TB_{BMO}I}}(n+2)^{1+\frac{1}{n+1}}|x-c|^{n+1}.\end{aligned}$$

Hence if $|x-c|<1$, $\lim_{n\to\infty}(n+2)^{1+\frac{1}{n+1}}|x-c|^{n+1}=0$, which implies that f is analytic on I. For each c, the radius of convergence r_c of the power series expansion about c is $\geq 1+|c|$, again by Lemma 1. Thus f is analytic in $\cup\{N_c : c\in I\}$, where $N_c=\{z : |z-c|<1+|c|\}$. But it is easily seen that $\cup N_c=E$, and this completes the proof.

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