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关于正关联BCK代数的一种扩张

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摘　要

在本文中，证明了每一个正关联BCK-代数X均可嵌入于具有条件(S)的正关联BCK-代数X^*，且X是X^*的一个子代数．特别地，当X是关联时，那么X^*也是关联的．

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The Condition (S) Extension of Positive Implicative BCK-Algebras *

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Abstract In this note, we prove that every positive implicative BCK-algebra X can be imbedded in a positive implicative BCK-algebra X^* with the condition (S), and X is a subalgebra of X^*. In particular, whenever X is implicative, so is X^*.

Keywords positive implicative BCK-algebra, implicative BCK- algebra, BCK-algebra with the condition (S), BCI-endomorphism, semigroup.

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Let a, b be elements of a BCI-algebra X with the condition (S). We denote the greatest element x satisfying $x * a \leq b$ by $a \circ b$, then $\circ$ is a binary operation of X and $(X, \circ, 0)$ is a commutative semigroup with identity 0.

This paper will prove that every positive implicative (or implicative) BCK- algebra can be imbedded in a positive implicative (or implicative) BCK-algebra with the condition (S). We call it the condition (S) extension.

It is easy to prove the following two propositions.

Proposition 1 *Let X be a BCI-algebra and $End(X)$ the set of all BCI-endomorphisms of X. Then $End(X)$ with respect to the composition of mappings is a semigroup with identity that is an identical mapping.*

Proposition 2 *Let X be a positive implicative BCK-algebra and $a \in X$. Put $f_a : x \mapsto x * a$. Then f_a is a BCI- endomorphism of X.*

We call f_a the right-hand multiplier of X and write S as the set consisting of all right-hand multipliers of X, that is,

$$S = \{f_a | a \in X\}.$$

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Proposition 3 *Let X be a positive implicative BCK-algebra. Then X satisfies the condition (S) if and only if S is a sub-semigroup of End(X).*

Proof Assume that X satisfies the condition (S). Put $f_a, f_b \in S$. Then, for any $x \in X$,

$$(f_a \cdot f_b)(x) = f_a(f_b(x)) = (x * b) * a = x * (b \circ a) = f_{b \circ a}(x),$$

so $f_a \cdot f_b = f_{b \circ a} \in S$ and hence S is a sub-semigroup of End(X).

Conversely assume that S is a sub-semigroup of End(X). Then, for any $a, b \in X$, there exists $c \in X$ such that $f_b \cdot f_a = f_c$, then

(i) $(c * a) * b = f_b(f_a(c)) = (f_b f_a)(c) = f_c(c) = c * c = 0$;

(ii) If $(x * a) * b = 0$ then

$$x * c = f_c(x) = (f_b f_a)(x) = f_b(f_a(x)) = (x * a) * b = 0,$$

this shows that c is the greatest element satisfying $x * a \leq b$, that is, X satisfies the condition (S).

In the proof of proposition 3, we obtain $f_a f_b = f_{b \circ a}$. Since X is commutative with respect to $\circ$, we have $f_{b \circ a} = f_{a \circ b}$, so $f_a f_b = f_b f_a$. Particularly, when $b = 0, f_a f_0 = f_{a \circ 0} = f_a$. This gives the following

Proposition 4 *Let X be a positive implicative BCK-algebra with the condition (S). Then the sub-semigroup S of End(X) is commutative and has an identity f_0.*

It is easy to see that

Proposition 5 *Let X be a positive implicative (or implicative) BCK-algebra. Define the following binary operation $*$ on S:*

$$f_a * f_b = f_{a * b}, \quad \text{for any} \quad a, b \in X.$$

Then

(1) $(S, *, f_0)$ is also a positive implicative (or implicative) BCK- algebra.

(2) X is isomorphic to S.

Note that S with respect to the operation of the composition of mappings might not be closed by the proposition 3, we consider the sub-semigroup S^* of End(X) generated by S. Firstly, we define

$$x * \prod_{i=1}^{n} a_i = (\cdots((x * a_1) * a_2) * \cdots) * a_n, x, a_i \in X, i = 1, 2, \cdots, n.$$

Then by induction, we can verify that

(I) $(x * \prod_{i=1}^{m} a_i) * \prod_{j=1}^{n} b_j = (x * \prod_{j=1}^{n} b_j) * \prod_{i=1}^{m} a_i$

in BCI-algebras.

(II) $(x * \prod_{i=1}^{m} a_i) * \prod_{j=1}^{n} (b_j * \prod_{i=1}^{m} a_i) = (x * \prod_{j=1}^{n} b_j) * \prod_{i=1}^{m} a_i$

in positive implicative BCK-algebras.

(III) $a_i * \prod_{j=1}^{m}(b_j * \prod_{k=1}^{n} a_k) = a_i, i = 1, 2, \cdots, n$

in implicative BCK-algebras.

Secondly, we define

$$\prod_{i=1}^{n} f_{a_i} = f_{a_1} f_{a_2} \cdots f_{a_n}, f_{a_i} \in S, i = 1, 2, \cdots, n.$$

Now we have

Theorem 6 *Let X be a positive implicative (or implicative) BCK-algebra and suppose that S^* is the sub-semigroup of $End(X)$ generated by S, that is,*

$$S^* = \{\prod_{i=1}^{n} f_{a_i} \mid f_{a_i} \in S, i = 1, 2, \cdots, n\}.$$

Define a binary operation on S^* as the following

$$(\prod_{i=1}^{m} f_{a_i}) * (\prod_{j=1}^{n} f_{b_j}) = \prod_{i=1}^{m} f_{a_i} * \prod_{j=1}^{n} b_j.$$

Then $(S^*, *, f_0)$ is a positive implicative (or implicative) BCK-algebra.

Proof (1) We have

$$\begin{aligned}
&(((\prod_i f_{a_i}) * (\prod_j f_{b_j})) * ((\prod_i f_{a_i}) * (\prod_k f_{c_k}))) * ((\prod_k f_{c_k}) * (\prod_j f_{b_j}))\\
&= ((\prod_i f_{a_i * \prod_j b_j}) * (\prod_i f_{a_i * \prod_k c_k})) * (\prod_k f_{c_k * \prod_j b_j})\\
&= (\prod_i f_{(a_i * \prod_j b_j) * \prod_l (a_l * \prod_k c_k)}) * (\prod_k f_{c_k * \prod_j b_j})\\
&= \prod_i f_{((a_i * \prod_j b_j) * \prod_l (a_l * \prod_k c_k)) * \prod_k (c_k * \prod_j b_j)}\\
&= f_0.
\end{aligned}$$

The last equality holds from

$$\begin{aligned}
&((a_i * \prod_j b_j) * \prod_1 (a_1 * \prod_k c_k)) * \prod_k (c_k * \prod_j b_j)\\
&= ((a_i * \prod_j b_j) * \prod_k (c_k * \prod_j b_j)) * \prod_l (a_l * \prod_k c_k) \quad \text{(by (I))}\\
&= ((a_i * \prod_k c_k) * \prod_j b_j) * \prod_l (a_l * \prod_k c_k) \quad \text{(by (II))}\\
&= ((a_i * \prod_k c_k) * \prod_l (a_l * \prod_k c_k) * \prod_j b_j \quad \text{(by (I))}\\
&= ((a_i * \prod_l a_l) * \prod_k c_k) * \prod_j b_j \quad \text{(by (II))}\\
&= (((a_i * a_i) * \prod_{l \neq i} a_l) * \prod_k c_k) * \prod_j b_j = 0.
\end{aligned}$$

(2) $(\prod_i f_{a_i}) * f_0 = \prod_i f_{a_i * 0} = \prod_i f_{a_i}$.

(3) $f_0 * (\prod_i f_{a_i}) = f_{0 * \prod_i a_i} = f_0$.

(4) If $(\prod_i f_{a_i}) * (\prod_j f_{b_j}) = (\prod_j f_{b_j}) * (\prod_i f_{a_i}) = f_0$, then, for any $i, j, a_i * \prod_k b_k = 0$, and $b_j * \prod_1 a_1 = 0$, thus, for any $x \in X$,

$$\begin{aligned}(\prod_i f_{a_i})(x) &= x * \prod_i a_i = (x * \prod_i a_i) * 0 = (x * \prod_i a_i) * \prod_j (b_j * \prod_i a_i) \\ &= (x * \prod_j b_j) * \prod_i a_i \quad \text{(by (II))} \\ &= (x * \prod_i a_i) * \prod_j b_j \quad \text{(by (I))} \\ &= (x * \prod_j b_j) * \prod_i (a_i * \prod_j b_j) \quad \text{(by (II))} \\ &= x * \prod_j b_j = (\prod_j f_{b_j})(x),\end{aligned}$$

and hence $\prod_i f_{a_i} = \prod_j f_{b_j}$.

(5) For any $\prod_i f_{a_i}, \prod_j f_{b_j} \in S^*$, obviously $(\prod_i f_{a_i})(\prod_j f_{b_j}) \in S^*$ and

$$\begin{aligned}&(((\prod_i f_{a_i})(\prod_j f_{b_j})) * (\prod_i f_{a_i})) * (\prod_j f_{b_j}) \\ &= ((\prod_i f_{a_i * \prod_k a_k})(\prod_j f_{b_j * \prod_i a_i})) * (\prod_j f_{b_j}) \\ &= (\prod_j f_{b_j * \prod_i a_i}) * (\prod_j f_{b_j}) \\ &= \prod_j f_{(b_j * \prod_i a_i) * \prod_l b_l} = f_0.\end{aligned}$$

Also, if $((\prod_k f_{c_k}) * (\prod_i f_{a_i})) * (\prod_j f_{b_j}) = f_0$, then $(c_k * \prod_i a_i) * \prod_j b_j = 0$ for any k, thus

$$(\prod_k f_{c_k}) * ((\prod_i f_{a_i})(\prod_j f_{b_j})) = \prod_k f_{(c_k * \prod_i a_i) * \prod_j b_j} = f_0,$$

that is, $(\prod_i f_{a_i})(\prod_j f_{b_j})$ is the greatest element satisfying

$$(\prod_k f_{c_k}) * (\prod_i f_{a_i}) \leq \prod_j f_{b_j}.$$

So far we have already proved that S^* is a BCK-algebra with the condition (S).

Now if X is positive implicative, then

$$\begin{aligned}((\prod_i f_{a_i}) * (\prod_j f_{b_j})) * (\prod_j f_{b_j}) &= (\prod_i f_{a_i * \prod_j b_j}) * (\prod_j f_{b_j}) \\ &= \prod_i f_{(a_i * \prod_j b_j) * \prod_j b_j} = \prod_i f_{a_i * \prod_j b_j} = (\prod_i f_{a_i}) * (\prod_j f_{b_j})\end{aligned}$$

and so S^* is also positive implicative. On the other hand, if X is implicative then

$$(\prod_i f_{a_i}) * ((\prod_j f_{b_j}) * (\prod_i f_{a_i})) = (\prod_i f_{a_i}) * (\prod_j f_{b_j * \prod_i a_i})$$
$$= \prod_i f_{(a_i * \prod_j (b_j * \prod_k a_k)} = \prod_i f_{a_i} \quad \text{(by (III))}$$

and so S^* is also implicative. This completes the proof.

Proposition 5 implies that $(S, *)$ is a subalgebra of $(S^*, *)$ and note that $(X, *) \cong (S, *)$ we immediately have

Theorem 7 *Every positive implicative (or implicative) BCK- algebra X can be imbedded in a positive implicative (or implicative) BCK- algebra X^* with the condition (S) and X is a subalgebra of X^*.*

Remark Extension of S to S^* in Theorem 6 is from the sub-semigroup of End(X) generated by S and therefore this extension is a minimal extension in the sense of condition (S).

Using Theorem 7, we can give another proof of one of the result of [6].

Proposition 8 ([6],Theorem 3) *Let X be a non-zero implicative BCK-algebra. Then X contains at least one maximal ideal.*

Proof We need the following proposition (see [7], corollary 5): Every non-zero implicative BCK-algebra with the condition (S) contains a maximal ideal M_a for any $a \neq 0$ such that $a \notin M_a$.

Now if X satisfies the condition (S), clearly X contains one maximal ideal. If X does not satisfy the condition (S), by Theorem 7, there exists an implicative BCK-algebra X^* with the condition (S) such that $X \subset X^*$. Put $a \neq 0 \in X$. Then there exists a maximal ideal M_a of X^* such that $a \notin M_a$. We prove that $M = M_a \cap X$ is a maximal ideal of X.

It is clear that M is a proper subset of X and $0 \in M$. If $x, y * x \in M$ and $y \in X$, since M_a is an ideal of X^* and $y \in X \subset X^*$, we have $y \in M_a$, thus $y \in M_a \cap X = M$ and therefore M is a proper ideal of X.

For any $b \in X - M$, let $N = \langle M, b \rangle$ be an ideal of X generated by M and b. Since X is positive implicative, $N = \{x \in X | x * b \in M\}$. If $b \in M_a$, then, by $b \in X, b \in M$ which contradicts $b \in X - M$. This contradiction implies that $b \notin M_a$. Since M_a is maximal ideal of X^* and X^* is positive implicative, we have $x * b \in M_a$ for any $x \in X$. On the other hand since X is a subalgebra of X^* and $x, b \in X, x * b \in X$. Thus $x * b \in M_a \cap X = M$ and note that $N = \{x \in X | x * b \in M\}$, we immediately have $N = X$. This proves that M is a maximal ideal of X.

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摘　要

在本文中，证明了每一个正关联BCK-代数X均可嵌入于具有条件(S)的正关联BCK-代数X^*，且X是X^*的一个子代数．特别地，当X是关联时，那么X^*也是关联的．