椭圆型方程广义解Liouville型定理的一个注

梁鋆廷　　　　梁学信
(中山大学数学系，广州510275)　(华侨大学数学系，泉州362011)

摘　要

在E^n考虑椭圆方程

$$\operatorname{div} \vec{A}(x,u,\nabla u)=B(x,u,\nabla u),$$

其中$\vec{A},B$满足如下的结构条件:

$$\xi\cdot A(x,u,\xi)\geq|\xi|^p,\qquad 1<p<n,$$
$$|\vec{A}(x,u,\xi)|\leq\kappa|\xi|^{p-1},\quad \kappa\geq 1,$$

$$|B(x,u,\xi)|\leq b(x)|\xi|^\gamma, b(x)\in L_\infty(E^n), p-1\leq\gamma\leq p-1+p/n.$$

本文证明如果广义整解$u\in W^1_{p,\mathrm{loc}}(E^n)\cap L_\alpha(E^n)$, 其中

$$\begin{cases}\alpha>0 \text{ 可以任意} & \text{当 } \gamma=p-1,\\ \alpha=n(\gamma+1-p)/(p-\gamma) & \text{当 } p-1<\gamma\leq p-1+\frac{p}{n},\end{cases}$$

那么$u\equiv 0$.

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A Note on Liouville Type Theorem to Generalized Solutions of Elliptic Equations*

Liang Xiting
(Dept. of Math., Zhongshan Univ, Guangzhou, 510275)

Liang Xuexin
(Dept. of Math., Huaqiao Univ, Quanzhou, 362011)

Abstract It is proved that if the generalized entire solution u of the equation (1) satisfies $u \in W^1_{p,\mathrm{loc}}(E^n) \cap L_\alpha(E^n)$ for an appropriate α, it must be zero.

Keywords elliptic equation, entire solution, Liouville theorem

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Consider on the n-dimensional Euclidean space E^n the following elliptic equation:

$$\operatorname{div} \underline{A}(x,u,\nabla u) = B(x,u,\nabla u), \tag{1}$$

where $\underline{A}(x,u,\xi)$ and $B(x,u,\xi)$ are defined on $E^n \times E^1 \times E^n$, measurable with respect to x for fixed u and ξ, continuous with respect to u and ξ for fixed x and satisfying the structural conditions:

$$\begin{aligned} &\xi \cdot \underline{A}(x,u,\xi) \geq |\xi|^p, \quad 1 < p < n \\ &|\underline{A}(x,u,\xi)| \leq \kappa |\xi|^{p-1}, \quad \kappa \geq 1 \end{aligned} \tag{2}$$

$$|B(x,u,\xi)| \leq b(x)|\xi|^\gamma, \quad b(x) \in L_\infty(E^n) \text{ and } p-1 \leq \gamma \leq p-1+p/n$$

respectively. As usual we call u a generalized solution of (1) if for any $B_r = \{|x| < r\}$, there hold $u \in W^1_p(B_r)$ and

$$\int_{B_r} \{\nabla v \underline{A}(x,u,\nabla u) + vB(x,u,\nabla u)\}dx = 0, \quad \forall v \in \overset{\circ}{W}{}^1_p(B_r) \tag{1)'$$

In the case $\gamma = p-1$ and under the supplementry assumption

$$|b(x)| = O(|x|^{-1}) \text{ as } |x| \to \infty.$$

We see from Liang[1] that if the generalized entire solution u of (1) is bounded globally on E^n, then it must be a constant. The same result is proved by Yu-Liang in [2] for a

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one-sided bounded solution. In Yu-Liang [3], other type Liouville theorem is given, namely, if a generalized solution of (1) satisfies $u \in L_\alpha(E^n)$ for an arbitaray $\alpha > 0$, then u must be trivial. The latter result was obtained early by T.Kilpeläinen in [4] for $\underline{A}(x,u,\xi) = \underline{A}(x,\xi)$ and $B(x,u,\xi) \equiv 0$ and

$$u \in W^1_{p,\mathrm{loc}}(E^n) \cap L_{p^*}(E^n), p^* = np/(n-p), 1 < p < n.$$

Without any restriction on $b(x)$ we now prove the following:

Theorem *Let the condition (2) be satisfied and u a generalized entire solution of (1). If $u \in L_\alpha(R^n)$ with*

$$\begin{array}{ll} \alpha > 0 \text{ may be arbitarary} & \text{as } \gamma = p-1, \\ \alpha = n(\gamma+1-p)/(p-\gamma) & \text{as } p-1 < \gamma \le p-1+p/n, \end{array} \tag{3}$$

the $u \equiv 0$.

Proof Let $k_0 > 0$ be arbitrary and $\theta = \theta(k_0) > 0$ to be determinated later. By the assumption on u we take $R = R(k_0) > 1$ large enough that

$$\int_{B_{5r}/B_r} |u|^\alpha dx \le \theta \text{ as } r \ge R. \tag{4}$$

Let r_0, r_1 satisfying $0 \le r_1 < r_0 \le r$ be arbitrary and $\varsigma(x) = \varsigma(|x|)$ a piecewise linear continuous function of $|x|$ satisfying

$$\varsigma(x) = \begin{cases} 0 \text{ as } |x| \le 2r - r_0 \text{ or } |x| \ge 4r + r_0, \\ 1 \text{ as } 2r - r_1 \le |x| \le 4r + r_1. \end{cases}$$

Then $|\nabla\varsigma(x)| \le (r_0 - r_1)^{-1}$. Let $k > 0$ and $u^+ = \max(u, 0)$. Take $v = \varsigma^p(u-k)^+ \in W^1_p(B_{5r})$ as a test function and insert it into $(1)'$ (in which B_r is replaced by B_{5r}) we get that

$$\int_{A(k,r_0)} \varsigma^p |\nabla u|^p dx \le C \int_{A(k,r_0)} \{(u-k)|\nabla\varsigma|\varsigma^{p-1}|\nabla u|^{p-1} + (u-k)\varsigma^p|\nabla u|^\gamma\} dx, \tag{5}$$

where $A(k, r_0) = (B_{4r+r_0}/B_{2r-r_0}) \cap \{u > k\}$ is the effective domain of the integrations and C is the constant depending only on n, p, κ and the norm of $b(x)$ in $L_\infty(E^n)$. It follows from (5) by using the Young inequality that

$$\int_{A(k,r_0)} \varsigma^p |\nabla u|^p dx \le C \int_{A(k,r_0)} \{|\frac{u-k}{r_0 - r_1}|^p + (u-k)^{\frac{p}{p-\gamma}}\} dx,$$

where the constant C is independent of k, r_0 and r_1 (the same is the following). From the above inequality and Sobolev imbedding theorem we get

$$\begin{aligned} (\int_{A(k,r_1)} |u-k|^{p^*} dx)^{p/p^*} &\le C \int_{A(k,r_0)} |\nabla(\varsigma(u-k)^+)|^p dx \\ &\le C \int_{A(k,r_0)} \{|\frac{u-k}{r_0-r_1}|^p + (u-k)^{\frac{p}{p-\gamma}}\} dx. \end{aligned} \tag{6}$$

We distinguish tree cases. (i) The case $p-1<\gamma<p-1+p/n$; (ii) The case $\gamma=p-1+p/n$; (iii) The case $\gamma=p-1$.

In the first case by the interpolation inequality we have

$$\int_{A(k,r_0)}|u-k|^{\frac{p}{p-\gamma}}dx\le(\int_{A(k,r_0)}|u-k|^{\alpha}dx)^{\frac{1-\lambda}{\alpha}\frac{p}{p-\gamma}}(\int_{A(k,r_0)}|u-k|^{p^*}dx)^{\frac{p}{p^*}\frac{\lambda}{p-\gamma}},\qquad(7)$$

$$(p-\gamma)/p=(1-\lambda)/\alpha+\lambda/p^* \text{ i.e., } \lambda=(1/\alpha-1+\gamma/p)/(1/\alpha-1/p^*).$$

The assumption $\alpha=n(\gamma+1-p)/(p-\gamma)$ implies

$$\lambda=p-\gamma \text{ and } \frac{1-\lambda}{\alpha}\frac{p}{p-\gamma}=p/n.$$

Combining with (6), (7) and (4) yields

$$\begin{aligned}&(\int_{A(k,r_1)}(u-k)^{p^*}dx)^{p/p^*}\\&\le C\{(r_0-r_1)^{-p}\int_{A(k,r_0)}(u-k)^p dx+\theta^{p/n}(\int_{A(k,r_0)}(u-k)^{p^*}dx)^{p/p^*}\}.\end{aligned}\qquad(8)$$

If we take θ such that $C\theta^{p/n}\le\frac{1}{2}$, then, in virure of the arbitrariness of r_0, r_1 we deduce from (8) (by the use of a lemma of Giaquinta-Giusti[5]) that

$$(\int_{A(k,r_1)}(u-k)^{p^*}dx)^{p/p^*}\le C(r_0-r_1)^{-p}\int_{A(k,r_0)}(u-k)^p dx,\quad\forall 0\le r_1<r_0\le r.\qquad(9)$$

If $\alpha\ge p$, it follows from (9) that

$$\begin{aligned}\int_{A(k,r_1)}(u-k)^{\alpha}dx&\le|A(k,r_1)|^{1-\alpha/p^*}(\int_{A(k,r_1)}(u-k)^{p^*}dx)^{\alpha/p^*}\\&\le C|A(k,r_0)|^{\alpha/n}(r_0-r_1)^{-\alpha}\int_{A(k,r_0)}(u-k)^{\alpha}dx.\end{aligned}\qquad(10)$$

If $0<\alpha<p$, by the interpolation inequality we get from (9) that

$$\begin{aligned}&(\int_{A(k,r_1)}(u-k)^{p^*}dx)^{p/p^*}\\&\le C(r_0-r_1)^{-p}(\int_{A(k,r_0)}(u-k)^{\alpha}dx)^{p(1-\lambda_1)/\alpha}(\int_{A(k,r_0)}(u-k)^{p^*}dx)^{\lambda_1 p/p^*}\\&\le\frac{1}{2}(\int_{A(k,r_0)}(u-k)^{p^*}dx)^{p/p^*}+C(r_0-r_1)^{-pn(\frac{1}{\alpha}-\frac{1}{p^*})}(\int_{A(k,r_0)}(u-k)^{\alpha}dx)^{\frac{p}{\alpha}},\end{aligned}\qquad(11)$$

where $\lambda_1=(1/\alpha-1/p)/(1/\alpha-1/p^*)\in(0,1)$.

For the same reason as above, the first term on the right hand side of (11) can be neglected, then, it follows

$$\int_{A(k,r_1)}(u-k)^{\alpha}dx\le C[|A(k,r_0)|(r_0-r_1)^{-n}]^{(1-\alpha/p^*)}\int_{A(k,r_0)}(u-k)^{\alpha}dx.\qquad(10)'$$

We now prove that (10) and (10)' imply

$$\mathrm{vrai}_{B_{4r}/B_{2r}} \max u \le 2k_0, \tag{12}$$

respectively. For this take for $m = 0, 1, 2, \cdots$

$$k_m = 2k_0 - k_0/2^m, r_m = r/2^m \text{ and } J_m = \int_{A(k_m, r_m)} (u - k_m)^\alpha dx.$$

Replace k by k_{m+1}, and r_0, r_1 by r_m, r_{m+1}, respectively, we have

$$\begin{aligned} J_{m+1} &\le C\left(\int_{A(k_{m+1}, r_m)} \left| \frac{u - k_m}{k_{m+1} - k_m} \right|^\alpha dx\right)^{\alpha/n} \left(\frac{2^{m+1}}{r}\right)^\alpha J_m \\ &\le C\left(\frac{2^{m+1}}{k_0}\right)^{\frac{\alpha^2}{n}} \left(\frac{2^{m+1}}{r}\right)^\alpha J_m^{1+\alpha/n}, \quad m = 0, 1, 2, \cdots. \end{aligned} \tag{13}$$

(4) implies $J_0 \le \theta$. Suppose we have proved

$$J_m \le \delta^m \theta. \tag{14}$$

Then, combining (13) with (14), we obtain

$$J_{m+1} \le J_m C 2^{\alpha(1+\alpha/n)} (\theta k_0^{-\alpha})^{\alpha/n} R^{-\alpha} (\delta^{\alpha/n} 2^{\alpha(1+\alpha/n)})^m.$$

Because of $R \ge 1$, if we take $\theta = \theta(k_0) > 0$ satisfying

$$C 2^{\alpha(1+\alpha/n)} (\theta k_0^{-\alpha})^{\alpha/n} \le \delta \text{ and } \delta^{\alpha/n} 2^{\alpha(1+\alpha/n)} = 1.$$

Then (14) holds also for $m+1$. By induction (14) holds for all positive integers. It follows that

$$0 = \lim_{m\to\infty} J_m = \int_{A(2k_0, 0)} (u - 2k_0)^\alpha dx = \int_{B_{4r}/B_{2r}} |(u - 2k_0)^+|^\alpha dx,$$

and then (12) holds. By the weak maximum principle for generalized solutions of elliptic equations (see Yu-Liang[6]) we have

$$\mathrm{vrai}_{B_{3r}} \max u^+ \le \mathrm{vrai}_{\partial B_{3r}} \max u^+ \le \mathrm{vrai}_{B_{4r}/B_{2r}} \max u^+ \le 2k_0. \tag{15}$$

From the proof above we see that the results hold as r increases. Therefore (15) implies $u^+ = 0$ on E^n owing to the arbitrariness of k_0 and r. Similarly, we can prove $(-u)^+ = 0$. Thus, the conclusion of the theorem is ture for the case of $p - 1 < \gamma < p - 1 + p/n$.

In the second case, $\gamma = p - 1 + p/n$, we have

$$p^* = p/(p - \gamma) = n(\gamma + 1 - p)/(p - \gamma) = \alpha.$$

It follows from (6) that

$$\begin{aligned} \left(\int_{A(k, r_1)} (u - k)^\alpha dx\right)^{p/\alpha} \le\ & C\{(r_0 - r_1)^{-p} |A(k, r_0)|^{1-p/\alpha} \left(\int_{A(k, r_0)} (u - k)^\alpha dx\right)^{p/\alpha} \\ &+ \int_{A(k, r_0)} (u - k)^\alpha dx\}. \end{aligned} \tag{16}$$

Now, replacing (13) we have

$$J_{m+1}^{p/\alpha} \leq C\left[\left(\frac{2^{m+1}}{r}\right)^p \left(\int_{A(k_{m+1},r_m)} \left|\frac{u-k_m}{k_{m+1}-k_m}\right|^\alpha dx\right)^{1-p/\alpha} J_m^{p/\alpha} + J_m\right]$$

$$\leq C\left[\left(\frac{2^{m+1}}{r}\right)^p \left(\frac{2^{m+1}}{k_0}\right)^{1-p/\alpha} + 1\right] J_m, \quad m = 0,1,2,\cdots. \tag{17}$$

On account of $r \geq R \geq 1$, from (17) we deduce again (12). The conclusion of the theorem is also ture for $\gamma = p - 1 + p/n$.

Finally we consider the case $\gamma = p - 1$. Let $\alpha \in (0,p)$. It follows from (6)

$$(\int_{A(k,r_1)} (u-k)^{p^*} dx)^{p/p^*}$$

$$\leq C[(r_0 - r_1)^{-p} + 1](\int_{A(k,r_0)} (u-k)^\alpha dx)^{(1-\lambda_1)p/\alpha} (\int_{A(k,r_0)} (u-k)^{p^*} dx)^{\lambda_1 p/p^*}$$

$$\leq \frac{1}{2}(\int_{A(k,r_0)} (u-k)^{p^*} dx)^{p/p^*} + C[(r_0 - r_1)^{-1} \tag{18}$$

$$+1]^{pn(1/\alpha - 1/p^*)}(\int_{A(k,r_0)} (u-k)^\alpha dx)^{p/\alpha}, \quad 0 \leq r_1 < r_0 \leq r, \tag{19}$$

where λ_1 is the same as in (11). In virtue of the arbitrariness of r_0 and r_1 (18) implies

$$(\int_{A(k,r_1)} (u-k)^{p^*} dx)^{p/p^*}$$

$$\leq C[(r_0 - r_1)^{-1} + 1]^{pn(1/\alpha - 1/p^*)}(\int_{A(k,r_0)} (u-k)^\alpha dx)^{p/\alpha}, \quad 0 \leq r_1 < r_0 \leq r.$$

Repeating the argument above we obtain (12) again. The conclusion of the theorem is also ture for $\gamma = p - 1$. Q. E. D.

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