(8) $\|x\| = r \Rightarrow \|Tx\| \le \|x\|, \|x\| = R \Rightarrow \|Tx\| \ge \|x\|$;
(9) $\|x\| = R \Rightarrow \|Tx\| \le \|x\|, \|x\| = r \Rightarrow \|Tx\| \ge \|x\|$.
Then T has a fixed point in $P_{r,R}$.

By the proof of the Theorem 7 in [4], we can obtain following result by Lemma 2.

Theorem 3 *Let X be the same as in Lemma 2 and $T : P_R \to X$ be semiclosed 1-set-contraction mapping satisfying (1), suppose, for some $\delta > 0$, one of following conditions holds:*
(10) $\|x\| = r \Rightarrow \|Tx\| \le \|x\|, \|x\| = R \Rightarrow \|Tx\| \ge (1+\delta)\|x\|$;
(11) $\|x\| = R \Rightarrow \|Tx\| \le \|x\|, \|x\| = r \Rightarrow \|Tx\| \ge (1+\delta)\|x\|$.
Then T has a fixed point in $P_{r,R}$.

Remark Since a mapping which maps $P_R I$ into the cone P must be weakly inward, we know that Theorem 3 improves Theorem 7 in [4].

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References

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[2] K.Deimling and Hu Shouchuan, *fixed points of weakly inward maps in conical shells*, Nonl. Anal., Vol.12, No.3(1988), 227-230.
[3] K.Deimling, *Nonlinear functional analysis*, Springer, Berlin, 1984.
[4] Li Guozhen, *The fixed point index and the fixed point theorems of 1-set-contraction maps*, Proc. Amer. Math. Soc., Vol.104(1988), 1163-1170.
[5] Li Guozhen, *A new fixed point theroem on demi-compact 1- set-contraction mappings*, Proc. Amer. Math. Soc., Vol.97, No.2(1986), 277- 280.
[6] Zhao Yichun, *Nonlinear functional analysis and its application*, High Eduction Press, China, 1990.

1- 集压缩映射的非零不动点

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摘 要

本文给出了1- 集压缩映射的一些新的非零不动点定理，它们推广和改进了[1,2,4,5] 中的某些重要定理.

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Positive Fixed Points of 1-set-contraction Mappings *

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Abstract In present paper, we establish some new positive fixed point Theorems for 1- set-contraction maps, which extend and improve the main results in [1,2,4,5].

Keywords Positive fixed point, 1-set-contraction Maps.

Classification AMS(1991) 47H10/CCL O177.91

1. Introduction and Preliminaries

Let X be a real Banach space, $P \subset X$ a cone, i.e., P is closed convex such that $tP \subset P$ for all $t \geq 0$ and $P \cap (-P) = \{0\}$. We denote the set $\{x \in P : \|x\| \leq R\}$ by P_R and the set $\{x \in P : r \leq \|x\| \leq R\}$ by $P_{r,R}$, where $0 < r < R$.

In [1,2,4,5], positive fixed point theorems were obtained for condesing mappings and 1-set-contraction mapings which map P_R into P. In this paper, we consider a more general mapping which is 1-set-contractive and maps P_R into the whole space X. The results in this paper extend and improve the main results in [1,2,4,5].

Let D be a closed subset of X, a mapping $T : D \to X$ is said to be weakly inward on D if $T_x \in I_D(x)$ for every $x \in D$, where $\bar{I}_D(x) = \overline{I_D(x)}$ and $\overline{I_D(x)} = \{x + t(y - x) : t \geq 0 \text{ and } y \in D\}$. In case D is a cone P, This simply becomes

$$(*) \qquad x \in \partial P, x^* \in P^* \text{ and } x^*(x) = 0 \Rightarrow x^*(Tx) \geq 0,$$

where $P^* = \{x^* \in X : x^* > 0 \text{ on } P\}$ and P denotes the bonary of $P, T : D \to X$ is called a semi-closed 1-set-contration mapping if, T is 1-set-contractive and $I - T$ is closed. T is said to be a demi-compact 1- set-contraction mapping if T is 1-set-contractive and semi-compact. For the referred concepts of the condensing, k-set-contraction and semi-compact map etc., see [3,6].

2. Main Results

At the begining of this section, we recall that

Theorem [1] *Let $T : P_R \to X$ be a condensing mapping, which satisfies*

(1) $x \in \partial P, \|x\| \leq R, x^* \in P$ *and* $x^*(x) = 0 \Rightarrow x^*(Tx) \geq 0$;

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(2) $Tx \neq \lambda x$ for $\|x\| = r$ for all $\lambda > 1$.
Then T has a fixed point in P_R.

Now we prove the following

Lemma 1 *Let $T : P_R \to X$ be a 1-set-contraction mapping satisfing (1) and (2), and (c) If $\{x_n\}$ is any sequence in P_R such that $x_n - Tx_n \to 0$ as $n \to \infty$, then there exists a point $x' \in P_R$ with $x' - TX' = 0$. Then T has a fixed point in P_R*

Proof Choose a seqence such that $0 < t_n < 1$ and $t_n \to 0$ as $n \to \infty$, and consider the mapping $T : P_R \to X$ defined by $T_n = t_n T$. Obviously, T_n is a t_n-set-contraction mapping and satisfieds (1) and (2). Hence from the above Theorem, there exists $x_n \in P_R$ such that $x_n = T_n x_n = t_n T x_n$ for each n. Since the sequence $\{x_n\}$ is bounded and $x_n - Tx_n = (1 - 1/t_n)x_n \to 0$ as $n \to \infty$. Therefore T has a fixed point in P_R by the condition (c). □

Theorem 1 *Let $T : P_R \to X$ be a semi-compact 1-set- contraction mapping satisfying (1) and (2), and*

(3) There exists $e \in P \backslash \{0\}$ such that $x - Tx \neq \lambda e$ for $\|x\| = r$ and $\lambda > 0$.

Then T has a fixed point in $P_{r,R}$.

Proof Let $\phi_n : [0, R] \to [0, \delta]$ be continuous such that $\phi_n(t) = 0$ for $t \geq r$ and $\phi_n(t) = \delta$ for $t \leq r - 1/n$ and large n with δ such that $\delta\|e\| > r + C$, where $C = \sup\{\|T_x\| : x \in P_R\}$. Let $T_n x = Tx + \phi_n(\|x\|)e$ for $x \in P_R$. It is easy to check that T_n is 1-set-contractive and satisfies (1) and (2). From the semi-compactness of T, we can see that T_n satisfies (c). By Lemma 1, there exists $x_n \in P_R$ with $T_n x_n = x_n$. From the choice of ϕ_n, we cannot have that $\|x_n\| \leq r - 1/n$. In fact, if $\|x_n\| \leq r - 1/n$, then $x_n = Tx_n + \delta e$ and $r + C \geq \delta\|e\|$, this contradicts the choice of δ. Assume that $r - 1/n < \|x_n\| < r$ for large n, since $\phi_n(\|x_n\|)$ is bounded and T is semi-compact, without loss of generality, we assume that $x_n - Tx_n = \phi_n(\|x_n\|)e \to \lambda e$, for some $\lambda \in [0, \delta]$ and $x_n \to x_0$ with $\|x_n\| = r$. Hence $x_n = Tx_n + \lambda e$ and therefore $\lambda = 0$ by (3). This completes the proof. □.

Exchanging condion (2) and (3) in Theorem 1, we have

Theorem 2 *Let $T : P_R \to X$ be demi-compact 1-set- contraction mapping satisfying (1) and*

(4) There exists $e \in P\backslash\{0\}$ such that $x - Tx \neq \lambda e$ for $\|x\| = R$ and $\lambda > 0$;

(5) $Tx \neq \lambda x$ for $\|x\| = r$ and $\lambda > 1$.

Then, T has a fixed point in $P_{r,R}$.

Proof Let $\phi(t) = \psi(t)r/t + (1 - \psi(t))R/t$ for $0 < t \leq R$, where $\psi : [0, R]$ is a continuous function, $\psi(t) = 0$ for $t \leq r, \psi(t) = 1$ for $t = R$ and is linear in between. Then we consider

$$T_0 x = \frac{T(\phi(\|x\|)x)_k}{\phi(\|x\|)} \quad \text{for} \quad x \in P_R.$$

We claim that T_0 satisfies (1), (2), (3). Furthermore, T_0 is semi- compact and 1-set-contractive. Indeed, for any sequence $\{x_n\} \subset P_R$ such that $x_n - T_0 x_n \to y$, we have

$$x_n - \frac{T(\phi(\|x_n\|)x_n)}{\phi(\|x_n\|)} \to y.$$

Since $\phi(\|x_n\|)$ is bounded and $r \leq \phi(\|x\|) \leq R$, without loss of generaility, we assume that $\phi(\|x_n\|) \to \lambda, (r \leq \lambda \leq R)$. hence $\phi(\|x_n\|)x_n - T(\phi(\|x_n\|)x_n) \to \lambda y$. Since T is semi-compact, we have a seqence $\{x_m\}$ of $\{x_n\}$ and x_0 such that $((\|x_m\|)x_m) \to x_0$. Thus, we have $x_m \to x_0/\lambda$, hence T_0 is Semi-compact in P_R.

In addition, for any bounded subset $B \subset P_R$, let

$$A_{m,i} = \{x \in P_R : (i-1)R/m < \|x\| \leq iR/m\},$$

the well-known properties of α imply that

$$\alpha(T_0 B) \leq \lim_{m\to\infty} \max \frac{(T(\phi(\|x\|)x) : x \in B \cap A_{m,i})}{\min\{\phi(\|x\|) : x \in B \cap A_{m,i}}.$$

From the definition of ϕ, we have $\min\{\phi(\|x\|) : x \in B \cap A_{m,i}\} = m$ for large m and $\alpha(T(\phi(\|x\|)x) : x \in B \cap A_{m,i})$ is bounded, hence

$$\begin{aligned} \lim_{m\to\infty} \frac{\alpha(T(\phi(\|x\|)x) : x \in B \cap A_{m,i})}{\min\{\phi(\|x\|)x) : x \in B \cap A_{m,i}\}} &= \lim_{m\to\infty} \alpha(T(\phi(\|x\|)x) : x \in B \cap A_{m,i})/m \\ &= 0 \leq \alpha(B). \end{aligned}$$

Since for $r_0 \in (0, R], \phi$ is uniform continuous in $[r_0, R]$, we have

$$\begin{aligned} \alpha(T_0 B) &\leq \max\{\overline{\lim_{m\to\infty}} \max_{2\leq i\leq m} \frac{\alpha(T(\phi(\|x\|)x) : x \in B \cap A_{m,i})}{\min(\phi(\|x\|).: x \in B \cap A_{m,i})} \alpha(B)\} \\ &\leq \max\{\lim_{m\to\infty} \max_{2\leq i\leq m} \frac{\max(\phi(\|x\|) : x \in B \cap A_{m,i})}{\min(\phi(\|x\|) : x \in B \cap A_{m,i})} \alpha(B \cap A_{m,i}), \alpha(B)\} \\ &\leq \max\{\alpha(B), \alpha(B)\} = \alpha(B). \end{aligned}$$

Hence T_0 is 1-set-contractive on P_R.

By Theorem 1, we have $x_0 \in P_{r,R}$ such that $x_0 = T_0 x_0$. Hence $T(\phi(\|x_0\|)x_0) = \phi(\|x_0\|)x_0$. Let $y_0 = \phi(\|x_0\|)x_0$, then $y_0 \in P_{r,R}$ and $Ty_0 = y_0$, which completes the proof. □.

From Theorem 1 and Theorem 2, we can prove the following result easily:

Corollary 1 *Let $T : P_R \to X$ be a semi-compact 1-set- contraction mapping satisfying (1) and*

(6) $\|x\| = r \Rightarrow Tx \not\geq x; \|x\| = R \Rightarrow Tx \not\leq x;$

or

(7) $\|x\| = R \Rightarrow Tx \not\geq x; \|x\| = r \Rightarrow Tx \not\leq x.$

Then T has a fixed point in $P_{r,R}$.

Remark Since a condensing mapping must be semi-compact 1-set- contraction mapping, Theorem 1 and Theorem 2 extend Theorem 1 in [1] and Theorem 3 in [2] respectively. Utilizing Corollary 1 and applying the proof of the lemma in [4], we can easily prove that

Lemma 2 *Let the norm $\|\cdot\|$ of X be increasing with respect to P and $T : P_R \to X$ be a k-set-contraction mapping $(0 < k < 1)$ which satisfies (1) and one of following conditions:*

(8) $\|x\| = r \Rightarrow \|Tx\| \le \|x\|, \|x\| = R \Rightarrow \|Tx\| \ge \|x\|$;
(9) $\|x\| = R \Rightarrow \|Tx\| \le \|x\|, \|x\| = r \Rightarrow \|Tx\| \ge \|x\|$.
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By the proof of the Theorem 7 in [4], we can obtain following result by Lemma 2.

Theorem 3 *Let X be the same as in Lemma 2 and $T : P_R \to X$ be semiclosed 1-set-contraction mapping satisfying (1), suppose, for some $\delta > 0$, one of following conditions holds:*
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[3] K.Deimling, *Nonlinear functional analysis*, Springer, Berlin, 1984.
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[5] Li Guozhen, *A new fixed point theroem on demi-compact 1- set-contraction mappings*, Proc. Amer. Math. Soc., Vol.97, No.2(1986), 277- 280.
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