then $Q \subset M$ and $P \cap Q = (0)$ (since if $P \cap Q \neq (0)$, then $P \cap Q$ is a power of some primary ideal and hence $P \cap Q = \sqrt{P \cap Q} \in \mathrm{Spec}R$, which implies $P = Q$). It is easily proved that R has only three prime ideals P, Q and M, where $P \cap Q = (0)$. By the same argument as above we can prove that each non-zero primary ideal belonging to P or Q is a power of P or Q. Therefore each non-zero ideal of R is a power of some minimal prime ideal or is a power of some primary ideal belonging to the maximal ideal.

The converse is obvious. This completes the proof. □

The following Corollary gives a new characterization of generalized primary rings and its proof is analogous to that of Corollary in [1].

Corollary A ring R is a generalized primary ring if and only if R is a pseudo primary ring and (0) is a primary ideal of R.

References

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关于伪准素环的注记

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摘 要

一个有单位元的交换环R称为伪准素环，如果R的每个非零理想都是某个准素理想之幂. 本文证明了环R是伪准素环当且仅当R是准素环或R是两个域的直和或R是至多具有三个素理想的一维局部环，并且每个非零理想或是某个极小素理想之幂或是某个属于极大理想的准素理想之幂.

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A Note on Pseudo Primary Rings *

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Abstract A commutative ring with identity R is called a pseudo primary ring if each non-zero ideal of R is a power of a primary ideal. In this note we prove that a ring R is a pseudo primary ring if and only if R is a primary ring or R is a direct sum of two fields or R is a one-dimensional local ring with at most three prime ideals in which each non-zero ideal is a power of a minimal prime ideal or is a power of some primary ideal belonging to the maximal ideal.

Keywords primary ideal, commutative ring.

Classification AMS(1991) 13A17/CCL O153.3

A commutative ring with identity R is called a pseudo primary ring if each non-zero ideal of R is a power of a primary ideal[1]. In this note we give a complete classification of all pseudo primary rings in terms of other well-known types of rings and a new characterization of generalized primary rings. Therefore we generalize the main results of [1]-[4].

The notations and terminology used here are the same as that of Atiyah and MacDonald[5]. A commutative ring with idently R is called a primary ring if $|\mathrm{Spec}R| = 1$. A commutative ring with identity R is called a generalized primary ring if each ideal of R is primary[3].

Our result is

Theorem *A ring R is a pseudo primary ring if and only if R is a primary ring or R is a direct sum of two fields or R is a one- dimensional local ring with at most three prime ideals in which each non-zero ideal is a power of a minimal prime ideal or is a power of some primary ideal belonging to the maximal ideal.*

Proof Let R be a pseudo primary ring. If R is not a local ring, then there are at least two maximal ideals M_1 and M_2. If $M_1 \cap M_2 \neq (0)$, then $M_1 \cap M_2$ is a power of some primary ideal. So we have

$$M_1 \cap M_2 = \sqrt{M_1} \cap \sqrt{M_2} = \sqrt{M_1 \cap M_2} \in \mathrm{Spec}R$$

which implies $M_1 = M_2$, a contradiction. Thus $M_1 \cap M_2 = (0)$. Obviously, $R = M_1 + M_2$. By Chinese Remainder Theorem $R \cong R/M_1 \oplus R/M_2$, that is , R is a direct sum of two

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fields. In the following we suppose that R is a looal ring. We shall first prove that each non-maximal prime ideal of R is principal. Let $P \neq (0) \in \mathrm{Spec}R, M \in \mathrm{Max}R$ and $P \subset M$. If $PM = P$, then for any $x \in M - P$ we have

$$\overline{P}\overline{M} = \overline{P},$$

where $\overline{P} = P/(x)P, \overline{M} = M/(x)P$. For any $p \neq 0 \in P$, we have

$$(0) \neq (p) \subseteq (p) + (x)P.$$

So there are a primary ideal Q and a positive integer n such that

$$(p) + (x)P = Q^n.$$

It is obvious that $\sqrt{Q} \subseteq P$. From $(x)P \subseteq Q$ and $x \notin P \supseteq \sqrt{Q}$ it follows that $P \subseteq Q$. Thus $P = Q$ and

$$[(p) + (x)P]/(x)P = P^n/(x)P = (\overline{P})^n = \overline{PM}(\overline{P})^{n-1} = [(p + (x)P]/(x)P \cdot \overline{M}.$$

By Nakayama's lemma we get

$$(p) + (x)P = (x)P$$

which implies $P \subseteq (x)P$ and $P = (x)P$. Now for $p \neq 0 \in P$, there are primary ideal N and a positive integer m such that $(p) = N^m$. It is easily seen that

$$N \subseteq P = (x)P \subseteq (x).$$

Thus there is ideal

$$A = \{a \in R | y = xa \text{ for some } y \in N\}$$

such that $N = (x)A$. Since $x \notin P \supseteq \sqrt{N}$ we have $A \subseteq N$ and hence $N = (x)N$. So

$$(p) = N^m = (x)NN^{m-1} = (x)(p) = (xp)$$

and therefore there is an $r \in R$ such that

$$p = xpr.$$

Since $x \in M, 1 - xr$ is unit of R. Thus we have $p = 0$, a contradiction. This shows $PM \subset P$. Take $p \in P - MP$ and $x \in M - P$. Then

$$(p) + (x)P = Q^n$$

for some primary iseal Q and some positive integer n. Obviously $\sqrt{Q} \subseteq P$. If $n > 1$, then

$$p \in Q^n \subseteq MP,$$

a contradiction. So we have $(p) + (x)P = Q$. From $(x)P \subseteq Q$ and $x \notin P \supseteq \sqrt{Q}$ it follows that

$$P \subseteq Q = (p) + (x)P \subseteq (p) + MP \subseteq P$$

and therefore $P = (p) + MP$. Thus in the ring $\overline{R} = R/(p)$ we have

$$P/(p) = M/(p) \cdot P/(p).$$

Since $R/(p)$ is still a local pseudo primary ring, we conclude $P/(p) = (0)$ by the foregoing proof. This shows $P = (p)$, that is, each non-maximal prime ideal of R is principal. Next we shall prove $\dim R \leq 1$. If $\dim R > 1$, then in R there is a strict ascending chain $P \subset N \subset M$ of proper prime ideals, where $M \in \mathrm{Max}R$. Let $P = (x)$ and $N = (y)$. Then $x = yz \in P$ for some $z \in R$. From $y \notin P$ it follows that

$$z \in P = (x).$$

So there is a $r \in R$ such that $z = xr$. Then

$$x(1 - yr) = 0.$$

Since $y \in M, 1 - yr$ is a unit of R. Thus $x = 0$, that is, $P = (0)$. Now it is easily proved that R has only three prime ideals (0), $N = (y)$ and M. Let A be a primary ideal belonging to N. Then there is a positive integer n such that $y^n \in A$. So

$$N^n \subseteq A \subseteq N.$$

If $A \neq N^n$, then there is a positive integer k such that $A \subseteq N^k$ but $A \not\subseteq N^{k+1}$. Take $d \in A - N^{k+1}$. Then there is a $e \in R$ such that $d = y^k e \in A$. Obviously,

$$e \notin (y) = N = \sqrt{A}.$$

So we have

$$y^k \in A \text{ and } A = N^k.$$

Thus any non- zero ideal contained in N is a power of N. In particular, take $q \in M - N$, then there is a positive integer m such that

$$(yq) = N^m = (y^m).$$

If $m > 1$, then there is an $f \in R$ such that $yq = y^m f$ and hence

$$q = y^{m-1} f \in N,$$

a contradiction if $m = 1$, then there is a $g \in R$ such that $y = yqg$. Since $q \in M, 1 - qg$ is a unit of R. So

$$N = (y) = (0),$$

again a contradiction. This shows $\dim R \leq 1$. Finally, if $\dim R = 0$, then R is a primary ring. If $\dim R = 1$, then there is a strict ascending chain $P \subset M$ of proper prime ideals of R. Obviously, $M \in \mathrm{Max}R$. If R has only prime ideals P and M, let A be a non-zero primary ideal belonging to P, then by the same argument as above we can prove that A is a power of P. If R has another prime ideal Q such that

$$Q \neq P \text{ and } Q \neq M,$$

then $Q \subset M$ and $P \cap Q = (0)$ (since if $P \cap Q \neq (0)$, then $P \cap Q$ is a power of some primary ideal and hence $P \cap Q = \sqrt{P \cap Q} \in \mathrm{Spec}R$, which implies $P = Q$). It is easily proved that R has only three prime ideals P, Q and M, where $P \cap Q = (0)$. By the same argument as above we can prove that each non-zero primary ideal belonging to P or Q is a power of P or Q. Therefore each non-zero ideal of R is a power of some minimal prime ideal or is a power of some primary ideal belonging to the maximal ideal.

The converse is obvious. This completes the proof. □

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