Journal of Mathematical Research & Exposition
Vol.15, No.3, 350-352, Aug. 1995

Essential Subgroups of group *

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Keywords group, essential subgroups.

Classification AMS(1991) 20D/CCL O152

In this note we define essential subgruops of a gruop. A subgroup H of a group G is said to be an essential subgroup of G if for every subgroup K of G we have $K \cap H = \langle e \rangle$. Here we obtain conditions for groups to contain essential subgroups.

Definition 1 *Let G be a group and H any subgroup of G. If for every subgroup K of G we have $H \cap K = e$ then we say H is an essential subgroup of G.*

Example 1 Let $S = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = e, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = p_1, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = p_2, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = p_3, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = p_4, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = p_5 \right\}$. The subgroups of S_3 are $\{1, p_1\} = H_1, \{1, p_2\} = H_2, \{1, p_3\} = H_3, \{1, p_4, p_5\} = H_4$.
Clearly every subgroup of S_3 is an essential subgroup of S_3.

Definition 2 *Let G be a group. H a normal subgroup of G. If for every normal subgroup K of G we have $H \cap K = \{e\}$. Then H is said to be a strongly essential subgroup of G.*

Example 2 Let S_3 be as in Example 1, H_4 be the only normal subgroup of S_3. Clearly H_4 is a strongly essential subgroup of S_3.

Definition 3 *Let G be a group. H a normal subgroup of G. If for every subgroup K of G we have $H \cap K = \{e\}$. Then we say H is a weakly essential subgroup of G.*

Definition 4 *Let G be a group. If every subgroup H of G is essential in G. Then G is called an essential group.*

Example 3 Example 1 above is an essential group.

Definition 5 *Let G be a group. If every normal subgroup H of G is strongly essential we say the group G is a strongly essential group.*

*Received Feb.15, 1993.

Example 4 Every abelian group G which is essential is also strongly essential.

Definition 6 *Let G be a group. If every normal subgroup H of G is a weakly essential subgroup of G then we say G is a weakly essential group.*

Proposition 7 *Let G be an abelian group. If G is weakly essential then it is strongly essential and vice versa.*

Proof Obvious from the fact that every subgroup is normal.

Proposition 8 *Let G be an abelian group. The following are equivalent.*

(i) G is weakly essential

(ii) G is essential

(iii) G is strongly essential.

Proof Obvious from the fact that every group is normal.

Theorem 9 *The permutation group $S_n, n > 3$ of degree n is not an essential group.*

Proof To prove S_n is not an essential group it is sufficient to find a subgroup H of S_n and a subgroup K of S_n such that $H \cap K \neq e$. Let

$$H = \left\{ \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \end{pmatrix} = e \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 3 & 2 & 4 & \cdots & n \end{pmatrix} = p_1, \right.$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 3 & 2 & 1 & 4 & \cdots & n \end{pmatrix} = p_2, \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & 1 & 3 & 4 & \cdots & n \end{pmatrix} = p_3,$$
$$\left. \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & 3 & 1 & 4 & \cdots & n \end{pmatrix} = p_4, \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 3 & 1 & 2 & 4 & \cdots & n \end{pmatrix} = p_5 \right\}$$

be a subgroup of $S_n, K = \langle e, p_1 \rangle$ is also a subgroup of S_n. We have $K \cap H = e, p_1 \neq \langle e \rangle$. Hence $S_n, n > 3$ is not an essential group.

Theorem 10 *S_3 is a essential group, strongly essential group and a weakly essential group.*

Proof Obviously from example 1, S_3 is strongly essential, weakly essential group and an essential group.

Theorem 11 *$S_{n,n} > 3$ is not a weakly essential group.*

Proof To prove $S_{n,n} > 3$ is not a weakly essential group we need only to show that for some normal subgroup H of S_n we have a subgroup K of S_n such that $K \cap H \neq 0$. Take A_n the alternating group to be the normal subgroup of S_n. Clearly A_n contains non-trivial subgroups. Let H be one such then $A_n \cap H = H$. So $S_{n,n} > 3$ is not a weakly essential group.

Theorem 12 *S_5 is not a strongly essential group.*

Proof Take in S_4 the alternating subgroup A_4 which is clearly normal. V the non cyclic group of order 4 is also normal in S_4 and $A_4 \cap V = V$; hence S_4 is not a strongly essential group.

Theorem 13 *$S_{n,n} \neq 4$ is a strongly essential group.*

Proof If $n \neq 4$ we know the alternating group A_n is a simple group and a unique normal subgroup of S_n. Hence $S_{n,n} \neq 4$ is a strongly essential group as it has no other normal subgroups.

Problem Is every torsion free non abelian group

(i) an essential group?

(ii) strongly essential group?

(iii) weakly essential group?

Reference

[1] M.Hall, *The Theory of Groups*, Macmillan, 1959.