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Generalized Lie Superalgebras of Cartan Type*

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Abstract Let F be a field of characteristic $p \neq 2$. In this paper, we define the generalized Lie superalgebra over F and prove the criterion of simplicity of a Z-graded generalized Lie superalgebra. We give the definition of the finite dimensional Cartan generalized Lie superalgebra $W(n)$ and prove the simplicity of $W(n)$. Finally, for Cartan generalized Lie superalgebras $S(n)$ and $H(n)$, we give the same result as for $W(n)$.

Keywords generalized Lie superalgebra, Grassmann algebra.
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1. Basic definitions and criterion of simplicity

Let F be a field of characteristic $p \neq 2$, s be an arbitrary positive integer. $Z_{2^s} = \{\bar{0}, \bar{1}, \cdots \overline{2^s - 1}\}$ is the residue class ring mod 2^s, we write Z_{2^s} as M. Let G be an algebra over F, G is called the generalized superalgebra over F if G can be decomposed into a direct sum of subspaces $G = \bigoplus_{a \in M} G_a$ and $G_\alpha G_\beta \subset G - \{\alpha + \beta\}$, for any $\alpha, \beta \in M$. Let x be a nonzero element of generalized superalgebra G, if $x \in G_\alpha, \alpha \in M$, we say the x is homogeneous of degress α and we write $\deg x = \alpha$. If $\alpha = \bar{n} \in M$, we write $(-1)^\alpha = (-1)^n$. Throughout what follows, if $\deg x$ occurs in an expression, then it is assumed that x is homogeneous.

The subalgebra (or idea) of a generalized superalgebra is the graded subalgebra (or idea).

Let $G = \bigoplus_{a \in M} G_a$ be a generalized superalgebra, we define an operation $\langle\ ,\ \rangle$ in G:

$$\langle a, b \rangle = ab - (-1)^{(\deg a)(\deg b)} ba, \tag{1.1}$$

(As G is M-graded, we only define $\langle , \rangle$ on homogeneous elements of G).

A generalized superalgebra G is called commutative if $\langle a, b \rangle = 0$ fot all $a, b \in G$ and associative if $(ab)c = a(bc)$ for all $a, b, c \in G$. For an associative generalized superalgebra, we

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have the following identity:

$$\langle a, bc\rangle = \langle a, b\rangle c + (-1)^{(\deg a)(\deg b)} b\langle a, c\rangle \tag{1.2}$$

Let $V = \bigoplus_{\alpha\in M} V_\alpha$ be an M-graded space, then $\mathrm{End}V = \bigoplus_{\alpha\in M}\mathrm{End}_\alpha V$ is associative generalized superalgebra, where $\mathrm{End}_\alpha V = \{a \in \mathrm{End}V \mid a(V_\beta) \subset V_{\alpha+\beta}$, for any $\beta \in M\}$.

Let $\Lambda(n)$ be the Grassmann algebra in n varibles $\xi_1, \cdots, \xi_n$, i.e., $\Lambda(n) = \{a\xi_{i_1} \wedge \xi_{i_2} \wedge \cdots \wedge \xi_{i_r} \mid a \in F, r \leqslant n, 0 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n\}$. Let $\deg(a\xi_{i_1}\Lambda\xi_{i_2}\Lambda\cdots\xi_{i_r}) = \bar{r}$, where $i_1, i_2, \cdots, i_r$ are different from each other, then $\Lambda(n) = \bigoplus_{\alpha\in M}\Lambda(n)_\alpha$ is a generalized superalgebra (if $n < 2^s - 1$, then $\Lambda(n)_\alpha = 0$, for $\alpha = \overline{n+1}, \overline{n+2}, \cdots, \overline{2^s - 1}$). We call $\Lambda(n)$ the Grassmann generalized superalgebra, it is commutative and associative.

Definition 1 *A generalized Lie superalgebra is a generalized superalgebra $G = \bigoplus_{\alpha\in M} G_\alpha$ with an operation $\langle , \rangle$ satisfying the following axiom:*

$$\langle a, b\rangle = -(-1)^{(\deg a)(\deg b)}\langle b, a\rangle \text{(graded skew-symmetry)},$$

$$\langle a, \langle b, c\rangle\rangle = \langle\langle a, b\rangle, c\rangle + (-1)^{(\deg a)(\deg b)}\langle b, \langle a, c\rangle\rangle \text{(graded Jacobi identity)}.$$

If $G = \bigoplus_{\alpha_M} G_\alpha$ is a generalized Lie superalgebra, then G_0 is an ordinary Lie algebra, the multiplication about $\langle , \rangle$ on the left by elements of G_0 determines structure of a G_0-module on $G_\alpha, \alpha \in M$.

Let $G = \bigoplus_{\alpha\in M} G_\alpha$ be an associative generalized superalgebra, then the operation (1.1) turns G into a generalized Lie superalgebra. In particular, the associative generalized superalgebra End$V$ is a generalized Lie superalgebra about the operation (1.1), we denote this as pl (V).

Let $G = \bigoplus_{\alpha\in M} G_\alpha$ and $G' = \bigoplus_{\alpha\in M} G'_\alpha$ be two generalized Lie superalgebras. The linear mapping φ from G into G' is called a homomorphism if φ preserves the operation $\langle , \rangle$ and $\varphi(G_\alpha) \subset G'_\alpha$. Similary, we have the definition of the ismorphism.

Let $G = \bigoplus_{\alpha\in M} G_\alpha$ be a generalized Lie superalgebra, $V = \bigoplus_{\alpha\in M} V_\alpha$ be an M-graded linear space. The homomorphism $\varphi: G \to \mathrm{pl}(V)$ is called the graded representation of G on V, or the representation of G on V. In this case, we also say that V is a graded G-module, simply G-module. The G-module V (or the representation of G on V) is called irreducible if V contains no nontrivial submodules. The homomorphism $ad: G \to \mathrm{pl}(G)$ is called the adjoint representation of G, where $adx(y) = \langle x, y\rangle$ for all $x, y \in G$.

Let $G = \bigoplus_{\alpha\in M} G_\alpha$ be a generalized superalgebra. G is called Z-graded if there exists a fimily $\{G_i \mid i \in Z\}$ of finite-demensional M-graded subspaces of G, such that $G = \bigoplus_{i\in Z} G_i$ and $G_i G_j \subset G_{i+j}$ for $i, j \in Z$. Above Z-grading is said to be consistent if $G_{\bar{j}} = \bigoplus_{i\in Z} G_{2^s i+j}, j = 0, 1, \cdots, 2^s - 1$. Clearly, the Z-homogeneous element must be the homogeneous element if the Z-grading of G is consistent.

Let $G = \bigoplus_{i\in Z} G_i$ be a Z-graded generalized Lie superalgebra, the G_0 is a generalized Lie superalgebra, it is an M-graded subalgebra of G and $\langle G_0, G_1\rangle \subset G_j$, $j \in Z$; therefore, the adjoint representation of G induces an M-graded representation of G_0 on $G_i (i \in Z)$.

A Z-graded generalized Lie superalgebra $G = \bigoplus_{i\in Z} G_i$ is called irreducible if the representation of G_0 on G_{-1} is irreducible.

A Z-graded generalized Lie superalgebra $G = \oplus_{i\in Z} G_i$ is called transitive if $\{x \in G_n \mid \langle x, G_{-1}\rangle = 0\} = 0$ for $n \geqslant 0$.

A generalized Lie superalgebra $G = \oplus_{\alpha\in M} G_\alpha$ is called simple if it contains no nontrivial ideals and $\langle G, G\rangle \neq 0$.

Theorem 1 *Let $G = \oplus_{n-1} G_n$ be a (not necessarily consistently) Z-graded Lie superalgebra and $G_1 \neq \{0\}$, if G satisfies the following conditions:*

(1) *G is transitive and irreducible;*

(2) *$\langle G_n, G_1\rangle = G_{n+1}$ for all $n \geqslant -1$.*

Then the generalized Lie superalgebra G is simple.

Proof Let J be a graded idea of G, and $J \neq \{0\}$. Clearly, $J \cap G_{-1}$ is a G_0-submodule of G_{-1}. For any $x \in J \cap G_{-1}$, we have $x \in J = \oplus_{\alpha\in M}(J \cap G_\alpha)$ and $x \in G_{-1} = \oplus_{\alpha\in M}(G_{-1} \cap G_\alpha)$, so we can assume $x = \sum_{\alpha\in M} x_\alpha$, where $x_\alpha \in J \cap G_\alpha$ and $x = \sum_{\alpha\in M} g_\alpha$, where $g_\alpha \in G_{-1} \cap G_\alpha$. Then $\sum_{\alpha\in M}(x_\alpha - g_\alpha) = 0$ and $x_\alpha = g_\alpha$ for any $\alpha \in M$. Therefore $J \cap G_{-1} = \oplus_{\alpha\in M}(J \cap G_{-1}) \cap G_\alpha)$, i. e., $J \cap G_{-1}$ is an M-graded G_0-submodule of G_{-1}. So $J \cap G_{-1} = \{0\}$ or $J \cap G_{-1} = G_{-1}$ by the irreducibility of G.

If $J \cap G_{-1} = \{0\}$, write $\overline{G}_i = \oplus_{j=-1}^{i} G_j$, then we have the smallest nonegative integer n such that $J \cap \overline{G}_n \neq \{0\}$. Let $0 \neq x \in J \cap \overline{G}_n$, then $\langle x, G_{-1}\rangle \subset \overline{G}_{n-1}$ and $\langle x, G_{-1}\rangle \subset J$, so $\langle x, G_{-1}\rangle \subset \overline{G}_{n-1} \cap J = \{0\}$. Let $x = \sum_{i=0}^{n} x_i$, where $x_i \in G_i$ and $x_n \neq 0$, then $\sum_{i=0}^{n}\langle x_i, G_{-1}\rangle = \{0\}$ and $\langle x_i, G_{-1}\rangle = \{0\}$, $i = 0, 1, \cdots, n$. This contradicts the transitivity of G. Therefore $J \cap G_{-1} = G_{-1}$ and $G_{-1} \subset J$. By the condition (2), we have $G_0 = \langle G_{-1}, G_1\rangle \subset J$, $G_1 = \langle G_0, G_1\rangle \subset J, \cdots, G_n \subset J$ for all $n \geqslant 1$, so $G = J$. □

2 Generalized Lie Superalgebra of Cartan Type

Let $G = \oplus_{\alpha\in M} G_\alpha$ be a generalized Lie superalgebra, $D \in \mathrm{End}_\alpha G$, $\alpha \in M$, D is called a derivation of degree α of G if $D(ab) = D(a)b + (-1)^{\alpha(\deg a)} aD(b)$. We denote by $\operatorname{der} G \subset \mathrm{End}_\alpha G$ the space of all derivations of degree α. Set $\operatorname{der} G = \oplus_{\alpha\in M} \operatorname{der}_\alpha G$. Then $\operatorname{der} G$ is a subagebra of the generalized Lie superalgebra pl (G). The element of $\operatorname{der} G$ is called the derivation of G.

Let $\widetilde{\Lambda}(n)$ be a free associative generalized superalgebra with the generators $\xi_1, \cdots, \xi_n$ whose M-grading is given by deg $(\xi_{i_1}\xi_{i_2}\cdots\xi_{i_r}) = \bar{r}$. Let I be the ideal of $\widetilde{\Lambda}(n)$ generated by all elements $\xi_i\xi_j + \xi_j\xi_i$, then $\widetilde{\Lambda}(n)/I \simeq \Lambda(n)$. For convenience, we denote the element $\xi_{i_1} \wedge \xi_{i_2} \wedge \cdots \wedge \xi_{i_r}$ of $\Lambda(n)$ by $\xi_{i_1}\xi_{i_2}\cdots\xi_{i_r}$ and the operation Λ of $\Lambda(n)$ by multiplication.

Lemma 1 *Let P, Q be homogeneous elements of $\widetilde{\Lambda}(n)$. Then $PQ - (-1)^{(\deg P)(\deg Q)} QP \in I$.*

Proof Let $(\deg P)(\deg Q) = \bar{k}$.

If k is an even number, write $\deg P = \bar{t}$, $\deg Q = \bar{r}$, then $\overline{tr} = \bar{k}$ and $2 \mid tr$. So there is an even number in $\{r,t\}$. If r is even, we may suppose $Q = \xi_{i_1},\cdots,\xi_{i_r}$, $P = \xi_{j_1},\cdots,\xi_{j_t}$. As $\xi_{j_r}\xi_{j_1} + \xi_{j_1}\xi_{j_r} = \langle\xi_{j_r},\xi_{j_1}\rangle \in I$, we obtain $QP = \xi_{j_1}\cdots\xi_{i_r}\xi_{j_1}\cdots\xi_{j_t} = -\xi_{j_1}\cdots\xi_{i_{r-1}}(\xi_{j_1}\xi_{j_r})\xi_{j_2}\cdots\xi_{j_t} + \sigma'$, where $\sigma' \in I$. Go on as above, we have: $QP = \xi_{j_1}\cdots\xi_{j_t}\xi_{i_1}\cdots\xi_{i_r} + \sigma = PQ + \sigma$, where $\sigma \in I$. Therefore: $PQ - (-1)^{(\deg P)(\deg Q)}QP = -\sigma \in I$. Similarly, if t is even, then $PQ = QP + \sigma$, and we have:

$$PQ - (-1)^{(\deg P)(\deg Q)}QP = \sigma \in I.$$

It is the same when k is an odd number.

Proposition 1 *For any homogeneous elements $P_1,P_2,\cdots,P_n \in \Lambda(n)$, there is one and only one derivation $D \in \operatorname{der}\Lambda(n)$ such that $D(\xi_i) = P_i, i = 1,\cdots,n$.*

Proof (i) Let E be a derivation of degree a of $\widetilde{\Lambda}(n)$. By lemma 1, $E(\xi_i\xi_j + \xi_j\xi_i) = (E(\xi_i)\xi_j + (-1)^a\xi_jE(\xi_i)) + (E(\xi_j)\xi_i + (-1)^a\xi_iE(\xi_j)) \in I$, therefore $E(I) \subset I$.

(ii) Let $\Phi: \widetilde{\Lambda}(n) \to \widetilde{\Lambda}(n)/I \simeq \Lambda(n)$ be the canonical homomorphism. Then there exist $\widetilde{P}_1,\cdots,\widetilde{P}_n \in \widetilde{\Lambda}(n)$, such that $\Phi(\widetilde{P}_i) = P_i, i = 1,\cdots,n$. We assert that there exists a derivation $\widetilde{D}$ of $\widetilde{\Lambda}(n)$, such that $\widetilde{D}(\xi_i) = \widetilde{P}_i$. In fact, we may assume $\widetilde{P}_1,\cdots,\widetilde{P}_{r_1} \in \widetilde{\Lambda}(n)_{\bar{0}}, \widetilde{P}_{r_1+1}, \cdots,\widetilde{P}_{r_2} \in \widetilde{\Lambda}(n)_{\bar{1}},\cdots,\widetilde{P}_{r_{2^s-1}},\cdots,\widetilde{P}_n \in \widetilde{\Lambda}(n)_{\overline{2^s-1}}$. Let $\widetilde{D}'_j$ be a linear mapping of $\widetilde{\Lambda}(n), j = 0,1,\cdots,2^s-1$, such that:

$$\widetilde{D}'_j(\xi_i) = \begin{cases} \widetilde{P}_i, & \text{if } i \in \{r_j+1,r_j+2,\cdots,r_{j+1}\}, \\ 0, & \text{if } i \in \{1,\cdots,n\}\backslash\{r_j+1,\cdots,r_{j+1}\}, \end{cases}$$

where $\widetilde{P}_{r_0} = \widetilde{P}_1, \widetilde{P}_{r_{2^s}} = \widetilde{P}_n$.

As $\widetilde{\Lambda}(n)$ is free, $\widetilde{D}'_0,\widetilde{D}'_1,\cdots,\widetilde{D}'_{2^s-1}$ can be extended derivations $\widetilde{D}_0,\widetilde{D}_1,\cdots,\widetilde{D}_{2^s-1}$ of degress $\bar{0},\bar{1},\cdots,\overline{2^s-2}$ of $\widetilde{\Lambda}(n)$ respectively. Therefore $\widetilde{D} = \sum_{i=0}^{2^s-1}\widetilde{D}_i$ is a derivation of $\widetilde{\Lambda}(n)$, and $\widetilde{D}(\xi_i) = \widetilde{P}_i$. It is clear that the derivation of $\widetilde{\Lambda}(n)$, which satisfies $\widetilde{D}(\xi_i) = \widetilde{P}_i, i = 1,\cdots,n$ is unique.

(iii) By (i), $\widetilde{D} \subset I$, so $\Phi(\widetilde{D}(I)) \subset \Phi(I) = 0$, and $I \subset \operatorname{Ker}(\Phi\widetilde{D})$. Therefore, there exists a unique derivation $D \in \operatorname{Der}\Lambda(n)$, such that the following diagram is commutative:

$$\begin{array}{ccc} \widetilde{\Lambda}(n) & \xrightarrow{\widetilde{D}} & \widetilde{\Lambda}(n) \\ \Phi\downarrow & & \downarrow\Phi \\ \Lambda(n) & \xrightarrow{D} & \Lambda(n) \end{array}$$

We have $D(\xi_i) = D\Phi(\xi_i) = \Phi\widetilde{D}(\xi_i) = \Phi(\widetilde{P}_i) = P_i, i = 1,2,\cdots,n$. □

Corollary 1 *For any $P_1,P_2,\cdots,P_n \in \Lambda(n)$, there exists a unique derivation $D \in \operatorname{Der}\Lambda(n)$, such that $D(\xi_i) = P_i$, $i = 1,2,\cdots,n$.*

Proof As $\Lambda(n) = \bigoplus_{a\in M}\Lambda(n)_a$, we may assume $P_i = \sum_{j=0}^{2^s-1}P_{ij}$, $i = 1,2,\cdots,n$, where $P_{ij} \in \Lambda(n)_j$. By proposition 1, for homogeneous elements $P_{1j},P_{2j},\cdots,P_{nj}, j = 0,1,\cdots,2^s-1$, there exists derivations D_j, such that $D_j(\xi_i) = P_{ij}, i = 1,2,\cdots,n$. Let $D = \sum_{j=0}^{2^s-1}D_j$, then

$D(\xi_i) = \sum_{j=0}^{2^n-1} D_j(\xi_i) = \sum_{j=0}^{2^n-1} P_{ij} = P_i,\ i = 1,\cdots,n$.

In particular, for $i = 1,2,\cdots,n$, there exist derivations D_i of $\Lambda(n)$, such that $D_i(\xi_j) = \delta_{ij},\ j = 1,2,\cdots,n$. We denote D_i by $\frac{\partial}{\partial \xi_i}$, $\mathrm{der}\Lambda(n)$ by $W(n)$. Let D be any element of $W(n)$. Then $D = \sum_{i=1}^{n} P_i \frac{\partial}{\partial \xi_i}$, where $P_i = D(\xi_i),\ i = 1,2,\cdots,n$.

Let $0 \leqslant k \leqslant n$, $\Lambda(n)_k = \langle a\xi_{i_1}\xi_{i_2}\cdots\xi_{i_k} \mid 0 \leqslant i_1 < \cdots < i_k \leqslant n, a \in F\rangle$. If $k > n$ or $k < 0$, set $\Lambda(n)_k = 0$, then $\Lambda(n) = \oplus_{k\in Z}\Lambda(n)_k$ is a generalized Lie superalgebra with the consistent Z-grading. The Z-grading of $\Lambda(n)$ induces a Z-grading of $W(n)$:

$$W(n) = \oplus_{k\in Z} W(n)_k,$$

where $W(n)_k = \{\Phi \in W(n) \mid \Phi(\Lambda(n)_j) \subset \Lambda(n)_{j+k}$, for any $j \in Z\}$.

As $W(n) = \{\sum_{i=1}^{n} P_i \frac{\partial}{\partial \xi_i} \mid P_i \in \Lambda(n)\}$, if $k \in \{-1,0,\cdots,n-1\}$, then $W(n)_k = \{\sum_{i=1}^{n} P_i \frac{\partial}{\partial \xi_i}\} \mid \deg P_i = k+1, i = 1,\cdots,n\}$; if $k \in Z\backslash\{-1,0,1,\cdots,n-1\}$, then $W(n)_k = \{0\}$. Therefore $W(n) = \bigoplus_{k=-1}^{n-1} W(n)_k$.

In particular, $W(n)_0$ is a Lie algebra, $W(n)_{-1} = \sum_{i=1}^{n} F \frac{\partial}{\partial \xi_i}$, so $\langle \frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \xi_j}\rangle = 0$, i.e.,

$$\frac{\partial}{\partial \xi_i}\frac{\partial}{\partial \xi_j} + \frac{\partial}{\partial \xi_j}\frac{\partial}{\partial \xi_i} = 0.$$

Theroem 2 (*a*) *The generalized Lie superalgebra $W(n)$ is transitive.*

(*b*) *$W(n)$ is irreducible.*

(*c*) *If $n \geqslant 2$, then $\langle W(n)_k, W(n)_1\rangle = W(n)_{k+1}$, for all $k \geqslant -1$.*

(*d*) *If $n \geqslant 2$, then $W(n)$ is simple.*

Proof (a) Let $P\frac{\partial}{\partial \xi_j} \in W(n)_k$, $k \geqslant 0$, if $\langle P\frac{\partial}{\partial \xi_j}, W(n)_{-1}\rangle = 0$, then $\langle P\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \xi_i}\rangle = 0, i = 1,2,\cdots,n$, by (1.1) and (2.1), we have:

$$P\frac{\partial}{\partial \xi_j}\frac{\partial}{\partial \xi_i} - (-1)^k P\frac{\partial P}{\partial \xi_i}\frac{\partial}{\partial \xi_j} + P\frac{\partial}{\partial \xi_i}\frac{\partial}{\partial \xi_j} = 0.$$

So $P\frac{\partial P}{\partial \xi_i}\frac{\partial}{\partial \xi_j} = 0$, and $P\frac{\partial P}{\partial \xi_i} = \frac{\partial P}{\partial \xi_i}\frac{\partial}{\partial \xi_j}(\xi_j) = 0,\ i = 1,2,\cdots,n$. Therefore $P \in \Lambda(n)_0$ and $P\frac{\partial}{\partial \xi_j} \in W(n)_{-1} \cap W(n)_k = 0$. The transitivity is proven.

(b) First, we prove $W(n)_0 \simeq \mathrm{gl}_n(V)$ (as Lie algebra). Let $V = \{v_1,\cdots,v_n\}$, $e_{ij}^* \in \mathrm{gl}_n(V)$ such that $e_{ij}^*(v_k) = \delta_{jk}v_i$. Then $e_{ij}^*(i,j = 1,\cdots,n)$ is the basis of $\mathrm{gl}_n(V)$, and $[e_{ij}^*, e_{kl}^*] = \delta_{jk}, e_{il}^* - \delta_{il}e_{kj}^*$. Let $\sigma: W(n)_0 \to \mathrm{gl}_n(V)$ be a linear mapping, such that $\sigma(\xi_i\frac{\partial}{\partial \xi_j}) = -e_{ji}^*$. By direct examination we see that σ is an isomorphism of Lie algebra.

Let $\Phi: W(n)_{-1} \to V$ be a linear mapping, such that $\Phi(\frac{\partial}{\partial \xi_i}) = v_i, i = 1,\cdots,n$. Then $\Phi(\langle \xi_i\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \xi_k}\rangle) = \Phi(-\delta_{ik}\frac{\partial}{\partial \xi_j}) = -\delta_{ik}v_j = -e_{ji}^*(v_k) = \sigma(\xi_i\frac{\partial}{\partial \xi_j})(\Phi\frac{\partial}{\partial \xi_k})$. So $W(n)_0$-module $W(n)_{-1}$

(about adjoint representation) is isomorphic to $gl(V)$ -module V (about canonical representation). Then we get that $W(n)_0$ -module $W(n)_{-1}$ is irreducible, and $W(n)$ is irreducible.

(c) By induction on k .

(d) By (a), (b), (c) and Theorem 1. □

Imitating 3 of [1], let $\omega = \theta\xi_1 \wedge \theta\xi_2 \wedge \cdots \wedge \theta\xi_n$ be volume form. Then $S(n) = \{D \in W(n) | D\omega = 0\}$ is a subalgebra of $W(n)$, and

$$S(n) = \langle \frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial \xi_j} + \frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial \xi_i} | f \in \Lambda(n),\ i,j = 1,\cdots,n \rangle.$$

Let $\omega_1 = \sum_{i=1}^{n}(d\xi_i)^2$ be Homiltonian form. Then $\widetilde{H}(n) = \{D \in W(n) | D\omega_1 = 0\}$ is subalgebra of $W(n)$. Let $H(n) = \langle \widetilde{H}(n), \widetilde{H}(n) \rangle$. Then $H(n) = \langle \sum_{i=1}^{n}\frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial \xi_j} | f \in \bigoplus_{j=0}^{n-1}\Lambda(n)_j,$ $i = 1,\cdots,n \rangle$ and $S(n) = \bigoplus_{k=-1}^{n-2}S(n)_k, H(n) = \bigoplus_{k=-1}^{n-3}H(n)_k$ are Z -graded generalized Lie superalgebra. Imitating Theorem 2, we obtain the following theroem: **Theorem 3** *Let $n > 3$. Then*

(a) Both $S(n)$ and $H(n)$ are transitive.

(b) $S(n)_0$ -module $S(n)_{-1}$ is isomorphic to $sl(V)$ -module V ; $H(n)_0$ -module $H(n)_{-1}$ is isomorphic to $so(V)$ -module V ; therefore both $S(n)$ and $H(n)$ are irreducible.

(c) $\langle S(n)_k, S(n)_1 \rangle = S(n)_{k+1}, \langle H(n)_k, H(n)_1 \rangle = H(n)_{k+1}$, for all $k \geq -1$.

(d) $S(n)$ and $H(n)$ are simple.

$W(n)$, $S(n)$ and $H(n)$ are called Cartan generalized Lie superalgebras.

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Cartan型广义李超代数

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摘 要

设 F 是特征不为2的域. 本文定义了 F 上的广义李超代数，证明了 Z -阶化广义李超代数的单性准则。然后定义了有限维 Cartan 型广义李超代数 $W(n)$ ，证明了 $W(n)$ 的单性. 最后指出对 Cartan 型广义李超代数 $S(n)$ 与 $H(n)$ ，亦有与 $W(n)$ 相似的结果.