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On Construction of Ultranet*

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In the paper "An initial study of the use of ultranet to construct nonstandard models" (Kexue Tongbao, No. 6, 1985, 415—417) the author used ultranets to construct the hyperreal field $^*R$, this method is as follows. Let $D = (D, \geqslant)$ be a non-empty directed set and I an infinite set: A net on D is a mapping $S: D \to I$. Suppose that $S = \{S_d \mid d \in D\}$ is an ultranet and $A \subset I$. Let $\mathscr{U}_S = \{A \mid S \text{ is residual in } A\}$. Then for an infinite set X, the nonstandard extension $^*X$ may be defined by $^*X = X^I/\mathscr{U}_S$. In particular, $^*R = R^I/\mathscr{U}_S(I = N)$.

In the general case it is possible from the process above that for any infinite subset A of X, $^*A - A = \varnothing$. For example, $S' = \{S'_d = j \mid j \in I, d \in D\}$ is an ultranet. It is easy to see that $\mathscr{U}_{S'} = \{U \in \mathscr{P}(I) \mid S' \text{ is residual } U\} = \{U \in \mathscr{P}(I) \mid j \in U\}$. Thus $\mathscr{U}_{S'}$ is a principal ultrafilter on I, and so the construction of $^*A$ collapses, i.e., $^*A = A^I/\mathscr{U}_{S'} \cong A$.

It is well-known from set-theoretic topology that $\mathscr{U}_S$ is the ultrafilter derived from the ultranet S. Thus, by a basic fact about the ultrafilter, we have the following result:

Theorem 1 *$^*A - A \neq \varnothing$ if and only if there exists a countable partition $\{I_n\}_{n \in N}$ of I such that for any $d \in D$ and $n \in N$ there is a $d' \in D$ so that $S_{d'} \notin I_n$ if $d' \geqslant d$.*

Proof Assertion follows from the equivalent condition that for every $n \in N$, $I_n \notin \mathscr{U}_S$.

We now prove the existence of such ultranet.

Theorem 2 *Notation is as indicated above. There is an ultranet $S = \{S_d \mid d \in D\}$ satisfying Theorem 1.*

Proof Since the family $\mathscr{F} = \{I'_n\}_{n \in N}$ of subsets of I has the finite intersection property, there is an ultrafilter $\mathscr{E}$ over I such $\mathscr{F} \subseteq \mathscr{E}$, and so $I'_n \in \mathscr{E}$ for all $n \in N$. Let $\mathscr{E} = \{E_d \mid \in D\}$, where D is an index set and we can regard D as a directed set, defining that $d \geqslant d'$ if and only if $E_d \subseteq E_{d'}$ (Assume that $E_d \neq E_{d'}$ if $d \neq d'$). We define a net $S = \{S_d \mid d \in D\}$ on D by $S_d \in E_d$ for every $d \in D$. Clearly S is an ultranet, and satisfies the condition of theorem 1. In fact, for any $d \in D$ and $n \in N$, if $d_1 \geqslant d$ then $E_{d_1} \subseteq E_d$, and $I'_n \in \mathscr{E}$. Since $\mathscr{E}$ is closed under the formation of finite intersection, let $E_{d_1} \cap I'_n = E_{d'}$. Thus $E_{d_1} \subseteq E_{d'}$ and $d \geqslant d_1 \geqslant d'$. We also have $S_{d'} \in E_{d'} \subseteq I'_n$, i.e., $S_{d'} \notin I_n$.

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