tends to the standardized normal distribution, therefore

$$-u_\alpha - \frac{S_1 - N_1\theta_1}{\sqrt{N_1\underline{\theta}_1(1-\underline{\theta}_1)}} + \sqrt{\frac{N_1}{\underline{\theta}_1(1-\underline{\theta}_1)}}(\underline{\theta}_1 - \theta_1) \to 0 \quad \text{wp } 1,$$

which shows that the theorem holds. □

References

[1] H.F.Martz and R.A.Waller, *Bayesian Reliability Analysis*, John Wiley & Sons, 1982.

[2] Zheng Zhongguo, *The Virtual System Method for Estimation of Parameter in System* , Tree. Tech. Report, Peking University, 1991.

[3] Zheng Zhongguo, *Virtual System Method and its Application* , Tech. Report, Peking University, 1991.

[4] Zheng Zhongguo and Fang Kaitai, *Large Sample Property of Estimates of Multiparameters in a System Tree* , Tech. Report of Hong Kong Baptist College, Dept. of Math., No.13, 1992.

[5] N.R.Mann et al, *Approximately Optimum confidence bounds for system reliability based on component data* , Techaometrics, **16:3**(1974), 335-347.

[6] Sheng Zhou and Fang Dayin, *The Comparation of the Baysion Confidence Limit of the Reliability of the Binomal Series System Which Prior Distribuction Is B(0,0) with Classical Accurate Solution and the Result of AO Method (Chinese)* , Mathematical Statistics and Applied Probability (Supplymentary Issure), 1987, 161-164.

系统树中的近似 Bayes 方法

郑忠国　戴　明[†]　石　坚[‡]

(北京大学概率统计系，100871)

摘　要

本文利用近似 Bayes 方法对一个具有树形结构的成败型系统的可靠性进行估计. 本文证明了利用近似 Bayes 方法对系统的可靠性参数的估计，其点估计在 Fisher 意义下是渐近有效的，其相应的置信下限也是 Fisher 意义下渐近有效的.

[†]现在工作单位：北京理工大学.

[‡]现在工作单位：中国科学院系统科学研究所.

Journal of Mathematical Research & Exposition
Vol.16, No.1, 1-12, Feb. 1996

Approximation Bayes Method in System Tree *

Zheng Zhongguo *Dai Ming* *Shi Jian*
(Department of Probability and Statistics, Peking University, Beijing, 100871)

Abstract In this paper, an approximate Bayes method is employed to estimate the reliability of a system tree. It is proved that the estimators set by approximate Bayes method is efficient in Fisher's sense. Moreover, the lower confidence limit of the reliability of a system tree set by approximate Bayes method, which is efficient in Fisher's sense, is investigated.

Keywords system tree, Bayes method, prior distribution, posterior distribution, conjugate distribution, asymptotical efficiency.

Classification AMS(1991) 62F15/CCL O213.2

1 Introduction and Main Results

In practive, a system of a machine is usually divided into several subsystems, and these subsystems again can be divided into several subsystem. $\cdots$. Finally, a system tree is formed. An example of a system tree is explained in Figure 1.1.

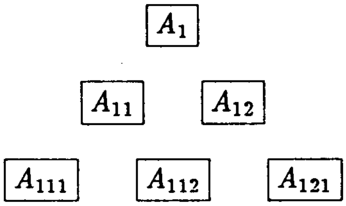

Figure 1.1

In Figure 1.1, system A_1 is divide into subsystems A_{11} and A_{12} where subsystem A_{11} is again divided into A_{111} and A_{112} and A_{12} has only one system A_{121} as its subsystem. In this paper, we denote the system tree by $\mathcal{A} = \{A_m, m \in M\}$, where M is a finite set of indices, satisfying

(1) $m = (1) \in M$;

(2) if $(i_1, \cdots, i_k) \in M$, then $i_1 = 1$;

*Received Jun.14, 1993.Supported parially by the NSFC and the Doctoral Program Fundation.

(3) if $(i_1,\cdots,i_k)\in M$, then $(i_1,\cdots,i_{k-1})\in M$, and $(i_1,\cdots,i_{k-1},i)\in M$,

$i=1,\cdots,i_{k-1}$,

where $i_1,\cdots,i_k$ are positive integers. $A_{\tilde{m}}$ is said to be the son of A_m, if $m=(i_1,\cdots,i_k)$ and $\tilde{m}=(i_1,\cdots,i_k,i_{k+1})$.$A_m$ is said to be the last generation subsystem of the system tree $\mathcal{A}$ if it has no som in $\mathcal{A}$. In Figure 1.1, A_{11} and A_{12} are the sons of A_1; A_{111}, A_{112} and A_{121} are the last generation subsystems of the system tree.

Denote

$$M(m)=\{\tilde{m}:A_{\tilde{m}} \text{ is the son of } A_m\},$$
$$M_0=\{m:A_m \text{ is the last generation subsystem of } \mathcal{A}\}.$$

Let θ_m be the reliability of A_m, i.e., the probability that A_m will work perfectly. For the relations between θ_m's, we have

Condition 1.1 *Suppose that $m\notin M_0$, and $M(m)=\{\tilde{m}_1,\cdots,\tilde{m}_l\}$, then*

$$\theta_m=\theta_m(\theta_{\tilde{m}_1},\cdots,\theta_{\tilde{m}_l}),$$

where the function $\theta_m(\theta_{\tilde{m}_1},\cdots,\theta_{\tilde{m}_l})$ is a known function with $\sum_{\tilde{m}\in M(m)}(\frac{\partial\theta_m}{\partial\theta_{\tilde{m}}})^2\neq 0$ and bounded second partial derivatives.

If the system structure is the series structure, then the function θ_m has the form of

$$\theta_m=\prod_{i=1}^{l}\theta_{\tilde{m}_i}.$$

It is easy to know that among the parmeters $\theta_m, m\in M$, the independent parameters are $\{\theta_m, m\in M_0\}$. Suppose that for every subsytem A_m, we have pass-failure data $\{n_m,s_m\}$ where n_m is the number of trials and s_n is the number of successed. Assume that $\{s_m, m\in M\}$ are independent to each other.

Condition 1.2 *For every $m,m'\in M$, there exists δ satisfying $0\le\delta\le\frac{1}{2}$ such that the following holds as $N=\min(n_m,m\in M)\to\infty$*

$$\lim_{N\to\infty}\frac{n_m^{1+\delta}}{n_{m'}}=+\infty.$$

The main purpose of our task is to assess the reliability θ_1 of the system A_1 (the machine itself) from the whole data set $\{(n_m,s_m),m\in M\}$. In [3], an iterative method which we called the virtual system method is presented and the efficiency of the estimate and the lower-bound of θ_1 is proved. In this paper, we will deal with the approximate Baysian method, which was introduced by N.R.Mann at [5] (also see [6]).

Let $\prod_{m\in M_0}p_m(t_m)$ be the prior density of $(\theta_m, m\in M_0)$. We calculate the approximate posterior distribution iteratively through the following steps.

(1) For every $m\in M_0$, the approximate posterior distribution $\tilde{p}_m$ is the beta distribution with parameters S_m and N_m-S_m+1.

$$\beta(S_m,N_m-S_m+1),$$

where

$$N_m = n_m + \tilde{n}_m, \quad S_m = s_m + \tilde{m}, \tag{1.1}$$

$$\tilde{n}_m = \frac{m_1(m) - m_2(m)}{m_2(m) - m_1^2(m)} - 1, \tag{1.2}$$

$$\tilde{s}_m = \frac{m_1(m) - m_2(m)}{m_2(m) - m_1^2(m)} m_1(m), \tag{1.3}$$

$$m_1(m) = \int_0^1 t p_m(t) dt, \tag{1.4}$$

$$m_2(m) = \int_0^1 t^2 p_m(t) dt. \tag{1.5}$$

(2) For $m \notin M_0$, the approximate posterior distribuction is

$$\beta(S_m, N_m - S_m + 1),$$

where N_m and S_m are given by (1.2)-(1.3), but $m_1(m)$ and $m_2(m)$ are given by

$$m_1(m) = \int \cdots \int \theta_m(t_{\tilde{m}}, \tilde{m} \in M(m)) \prod_{\tilde{m} \in M(m)} \tilde{p}_{\tilde{m}}(t_{\tilde{m}}) dt_{\tilde{m}}, \tag{1.6}$$

$$m_2(m) = \int \cdots \int \theta_m^2(t_{\tilde{m}}, \tilde{m} \in M(m)) \prod_{\tilde{m} \in M(m)} \tilde{p}_{\tilde{m}}(t_{\tilde{m}}) dt_{\tilde{m}}. \tag{1.7}$$

In the above, $\tilde{p}_{\tilde{m}}(\cdot)$ is the density function of the approximate posterior distribuction $\beta(S_{\tilde{m}}, N_{\tilde{m}} - S_{\tilde{m}} + 1)$.

Definition 1.1 *Let $\underline{\theta}_1$ be the solution of the following equation*

$$\frac{\int_0^{\underline{\theta}_1} \theta^{S_1 - 1}(1-\theta)^{N_1 - S_1} d\theta}{\int_0^1 \theta^{S_1 - 1}(1-\theta)^{N_1 - S_1} d\theta} = \alpha, \tag{1.8}$$

and

$$\hat{\theta}_1 = \frac{S_1}{N_1 + 1}. \tag{1.9}$$

In Definition 1.1, $\underline{\theta}_1$ stands for the lower confidence limit of θ_1 with the designed level $1 - \alpha$, and $\hat{\theta}_1$ for the point estimator of θ_1.

The information matrix I of the data $((n_m, s_m), m \in M)$ is given by

$$I = \sum_{m \in M} \frac{\partial \theta_m}{\partial \theta} \frac{n_m}{\theta_m(1-\theta_m)} \frac{\partial \theta_m}{\partial \theta^\tau}, \tag{1.10}$$

where $\theta = (\theta_m, m \in M_0)^\tau$ is the column vector with components $\theta_m, m \in M_0$. It is easy to know that I is a matrix valued function of the parameters $\{\theta_m, m \in M_0\}$. In this paper, we will investigate the efficiency of $\hat{\theta}_1$ and $\underline{\theta}_1$. The following two theorems show that they are efficient in the Fisher's sense.

Theorem 1.1 *Let $\{\theta_m, m \in M\}$ be the set of the parameters of the system tree, and $(n_m, s_m), (m \in M)$ be the data set of the whole tree. Let $\hat{\theta}_1$ be the point estimator of θ_1 defined by (1.9). Under the Conditions 1.1 and 1.2 as $\min\{n_m, m \in M\} \to \infty$, the follwing holds*

$$\left(\frac{\partial\theta_1}{\partial\theta^\tau} I^{-1} \frac{\partial\theta_1}{\partial\theta}\right)^{-\frac{1}{2}} (\hat{\theta}_1 - \theta_1) \xrightarrow{d} \mathcal{N}(0,1). \tag{1.11}$$

Theorem 1.2 *Under the conditions of Theorem 2.1, the following holds:*

$$\left(\frac{\partial\theta_1}{\partial\theta^\tau} I^{-1} \frac{\partial\theta_1}{\partial\theta}\right)^{-\frac{1}{2}} (\underline{\theta}_1 - \theta_1) \xrightarrow{d} \mathcal{N}(u_\alpha, 1). \tag{1.12}$$

where $\underline{\theta}_1$ is the approximate Baysian lower confidence limit of θ_1 defined by (1.8), and u_α is the α quantile of the standardized normal distribution function.

2 Proofs

Lemma 2.1 *Let the true parameters of the system tree be $\{\theta_m, m \in M\}$. Suppose that S_m and N_m are defined by (1.1)–(1.7). Then as $N = \min\{n_m : n_{\tilde{m}}, \tilde{m} \in M(m)\} \to \infty$, the following holds,*

$$\left|\frac{S_m}{N_m + 1} - \theta_m\right| = O_p\left(\frac{1}{\sqrt{N}}\right), \tag{2.1}$$

$$\int_0^1 (t - \theta_m)^4 \tilde{p}_m(t)dt = O_p\left(\frac{1}{N^2}\right). \tag{2.2}$$

Proof We prove this lemma by induction. When $m \in M_0$, since the quantities $\tilde{n}_m$ and $\tilde{s}_m$ are constants, (2.1) and (2.2) can be obtained by direct calculation. Now suppose that $m \in M \backslash M_0$ and for every $\tilde{m} \in M(m)$. (2.1) and (2.2) hlod. Then from (1.2), (1.3), we have

$$\begin{aligned}\left|\frac{S_m}{N_m+1} - \theta_m\right| &= \left|\frac{n_m}{N_m+1}\left(\frac{s_m}{n_m} - \theta_m\right) + \frac{\tilde{n}_m + 1}{N_m + 1}\left(\frac{\tilde{s}_m}{\tilde{n}_m + 1} - \theta_m\right)\right| \\ &\le \left|\frac{s_m}{n_m} - \theta_m\right| + \left|\frac{\tilde{s}_m}{\tilde{n}_m} - \theta_m\right| \\ &= \left|\frac{s_m}{n_m} - \theta_m\right| + |m_1(m) - \theta_m|, \end{aligned} \tag{2.3}$$

where $m_1(m)$ is given by (1.6).

From (1.6) and Taylor expansion, we have

$$m_1(m) = \theta_m + \sum_{\tilde{m} \in M(m)} \int_0^1 \cdots \int_0^1 (t_{\tilde{m}} - \theta_{\tilde{m}}) \psi_{\tilde{m}} \prod_{m_1 \in M(m)} \tilde{p}_{m_1}(t_{m_1}) dt_{m_1}, \tag{2.4}$$

where ψ_{m_1} is the value of $\frac{\partial\theta_m}{\partial\theta_{\tilde{m}}}$ taking at $(\theta_{\tilde{m}}, \tilde{m} \in M(m)) + \xi_{m_1}(t_{\tilde{m}} - \theta_{\tilde{m}}, \tilde{m} \in M(m)), \xi_{m_1} \in [0,1]$. Since $\frac{\partial\theta_m}{\partial\theta_{\tilde{m}}}$ is bounded, we have

$$|m_1(m) - \theta_m)| \le c \sum_{\tilde{m} \in M(m)} \int_0^1 |(t - \theta_{\tilde{m}})| \tilde{p}_{\tilde{m}}(t)dt,$$

where c is a constant.

By induction, (2.2) implies that

$$\int_0^1 |(t-\theta_{\tilde{m}})\tilde{p}_{\tilde{m}}(t)dt = O_p(\frac{1}{\sqrt{N}}) \tag{2.5}$$

holds for all $\tilde{m} \in M(m)$. Therefore

$$|m_1(m) - \theta_m| = O_p(\frac{1}{\sqrt{N}}). \tag{2.6}$$

By the Central Limit Theorem (CLT) we obtain

$$\frac{s_m}{n_m} - \theta_m = O_p(\frac{1}{\sqrt{N}}). \tag{2.7}$$

(2.1) follows from (2.3), (2.6) and (2.7). Since $\tilde{p}_m$ is the density of beta distribution $\beta(S_m, N_m - S_m)$, the following holds:

$$\begin{aligned}
&\int_0^1 (t-\theta_m)^4 \tilde{p}_m(t)dt \\
&= \frac{S_m(S_m+1)(S_m+2)(S_m+3)}{(N_m+1)(N_m+2)(N_m+3)(N_m+4)} - 4\theta_m \frac{S_m(S_m+1)(S_m+2)}{(N_m+1)(N_m+2)(N_m+3)} \\
&+6\theta_m^2 \frac{S_m(S_m+1)}{(N_m+1)(N_m+2)} - 4\theta_m^3 \frac{S_m}{(N_m+1)} + \theta_m^4 \\
&= (\frac{S_m}{N_m+1} - \theta_m)^4 + \frac{6S_m(N_m+1-S_m)}{(N_m+1)^2(N_m+2)}(-\frac{N_m^2 S_m^2}{(N_m+1)^2(N_m+3)^2(N_m+4)} \\
&+(\frac{S_m N_m}{(N_m+1)(N_m+3)} - \theta_m)^2) + O_p(\frac{1}{N^2}),
\end{aligned}$$

which shows that (2.2) follows from (2.1). □

Lemma 2.2 *Suppose that $m \notin M_0$. As $N = \min\{n_{\tilde{m}}, \tilde{m} \in M(m)\} \to \infty$, we have*

$$m_1(m) - m_2(m) = \theta_m(1-\theta_m) + o_p(1), \tag{2.8}$$

where $m_1(m), m_2(m)$ are given by (1.6), (1.7).

Proof Let $t = (t_{\tilde{m}}, \tilde{m} \in M(m))$ be the vector in the space $(0,1)^{|M(m)|}$, where $|M(m)|$ is the number of the set $M(m)$. From Taylor expansion, we have

$$\theta_m(t) = \theta_m(\theta_{\tilde{m}}, \tilde{m} \in M(m)) + \sum_{\tilde{m} \in M(m)} \psi_{\tilde{m}}(t_{\tilde{m}} - \theta_{\tilde{m}}).$$

Substituting this expression into (1.6) and (1.7), we obtain

$$m_2(m) \quad = \quad \theta_m^2 + 2\theta_m \sum_{\tilde{m} \in M(m)} \int_0^1 \cdots \int_0^1 (t_{\tilde{m}} - \theta_{\tilde{m}})\psi_{\tilde{m}} \prod_{m_1 \in M} \tilde{p}_{m_1}(t_{m_1})dt_{m_1}$$

$$+\sum_{\tilde{m}\in M(m)}\int_0^1\cdots\int_0^1(t_{\tilde{m}}-\theta_{\tilde{m}})^2\psi_{\tilde{m}}^2\prod_{m_1\in M}\tilde{p}_{m_1}(t_{m_1})dt_{m_1}$$

$$+2\sum_{\substack{k,l\in M(m)\\k\neq l}}\int_0^1\cdots\int_0^1(t_k-\theta_k)(t_l-\theta_l)\psi_k\psi_l\prod_{m_1\in M}\tilde{p}_{m_1}(t_{m_1})dt_{m_1},$$

which together with (2.4), implies

$$m_1(m)-m_2(m)=\theta_m(1-\theta_m)+R,$$

where the reminder R is of order $o_p(1)$ as N tends to ∞. □

Lemma 2.3 *Suppose that $m\notin M_0$. Then under the conditions of Lemma 2.1, the following holds*

$$m_2(m)-m_1^2(m)=\sum_{\tilde{m}\in M(m)}(\frac{\partial\theta_m}{\partial\theta_{\tilde{m}}})^2\frac{1}{N_{\tilde{m}}+1}\theta_{\tilde{m}}(1-\theta_{\tilde{m}})+O_p(N_m^{-\frac{3}{2}}),\tag{2.9}$$

where $N_m=\min(N_{\tilde{m}},\tilde{m}\in M(m))$.

Proof From Taylor expansion, we have

$$\theta_m(t)=\theta_m+\sum_{\tilde{m}\in M(m)}(t_{\tilde{m}}-\theta_{\tilde{m}})\frac{\partial\theta_m}{\partial\theta_{\tilde{m}}}+\frac{1}{2}\sum_{k,l\in M(m)}\psi_{k,l}(t_k-\theta_k)(t_l-\theta_l),$$

where $\psi_{k,l}$ stands for the second partial derivative $\frac{\partial^2\theta_m}{\partial\theta_k\partial\theta_l}$ at $(\theta_{\tilde{m}},\tilde{m}\in M(m))+\xi(t_{\tilde{m}}-\theta_{\tilde{m}},\tilde{m}\in M(m)),\xi\in[0,1]$. Substituting the expresion of $\theta_m(t)$ into (1.6) and (1.7), we obtain

$$\begin{aligned}
m_1(m)&=\theta_m+\sum_{\tilde{m}\in M(m)}\frac{\partial\theta_m}{\partial\theta_{\tilde{m}}}\int_0^1(t_{\tilde{m}}-\theta_{\tilde{m}})\tilde{p}_{\tilde{m}}(t_{\tilde{m}})dt_{\tilde{m}}\\
&\quad+\frac{1}{2}\sum_{k,l\in M(m)}\int_0^1\cdots\int_0^1\psi_{k,l}(t_k-\theta_k)(t_l-\theta_l)\prod_{m_1\in M}\tilde{p}_{m_1}(t_{m_1})dt_{m_1},\\
m_2(m)&=\theta_m^2+\sum_{\tilde{m}\in M(m)}(\frac{\partial\theta_m}{\partial\theta_{\tilde{m}}})^2\int_0^1(t_{\tilde{m}}-\theta_{\tilde{m}})^2\tilde{p}_{\tilde{m}}(t_{\tilde{m}})dt_{\tilde{m}}\\
&\quad+2\theta_m\sum_{\tilde{m}\in M(m)}\frac{\partial\theta_m}{\partial\theta_{\tilde{m}}}\int_0^1(t_{\tilde{m}}-\theta_{\tilde{m}})\tilde{p}_{\tilde{m}}(t_{\tilde{m}})dt_{\tilde{m}}\\
&\quad+\theta_m\sum_{k,l\in M(m)}\int_0^1\cdots\int_0^1\psi_{k,l}(t_k-\theta_k)(t_l-\theta_l)\prod_{m_1\in M}\tilde{p}_{m_1}(t_{m_1})dt_{m_1}\\
&\quad+2\sum_{\substack{k,l\in M(m)\\k\neq l}}\prod_{i=k,l}\frac{\partial\theta_m}{\partial\theta_i}\int_0^1(t_i-\theta_i)\tilde{p}_i(t_i)dt_i
\end{aligned}$$

$$+\frac{1}{4}\int_0^1\cdots\int_0^1(\sum_{k,l\in M(m)}\psi_{k,l}(t_k-\theta_k)(t_l-\theta_l))^2\prod_{m_1\in M}\tilde{p}_{m_1}(t_{m_1})dt_{m_1}$$

$$+\sum_{k,l,\tilde{m}\in M(m)}\frac{\partial\theta_m}{\partial\theta_{\tilde{m}}}\int_0^1\cdots\int_0^1\psi_{k,l}(t_k-\theta_k)(t_l-\theta_l)(t_{\tilde{m}}-\theta_{\tilde{m}})\prod_{m_1\in M}\tilde{p}_{m_1}(t_{m_1})dt_{m_1}.$$

From these two equalities, we have

$$\begin{aligned}
&m_2(m)-m_1^2(m)\\
&=\sum_{\tilde{m}\in M(m)}(\frac{\partial\theta_m}{\partial\theta_{\tilde{m}}})^2(\int_0^1(t_{\tilde{m}}-\theta_{\tilde{m}})^2\tilde{p}_{\tilde{m}}(t_{\tilde{m}})dt_{\tilde{m}}-(\int_0^1(t_{\tilde{m}}-\theta_{\tilde{m}})\tilde{p}_{\tilde{m}}(t_{\tilde{m}})dt_{\tilde{m}})^2)\\
&\quad+\frac{1}{4}(\int_0^1\cdots\int_0^1(\sum_{k,l\in M(m)}\psi_{k,l}(t_k-\theta_k)(t_k-\theta_k))^2\prod_{m_1\in M}\tilde{p}_{m_1}(t_{m_1})dt_{m_1}\\
&\quad-(\sum_{k,l\in M(m)}\int_0^1\cdots\int_0^1\psi_{k,l}(t_k-\theta_k)(t_l-\theta_l)\prod_{m_1\in M}\tilde{p}_{m_1}(t_{m_1})dt_{m_1})^2)\\
&\quad+\sum_{k,l,\tilde{m}\in M(m)}\frac{\partial\theta_m}{\partial\theta_{\tilde{m}}}\int_0^1\cdots\int_0^1\psi_{k,l}(t_k-\theta_k)(t_l-\theta_l)(t_{\tilde{m}}-\theta_{\tilde{m}}\\
&\quad-\int_0^1(t_{\tilde{m}}-\theta_{\tilde{m}})\tilde{p}_{\tilde{m}}(t_{\tilde{m}})dt_{\tilde{m}})\prod_{m_1\in M}\tilde{p}_{m_1}(t_{m_1})dt_{m_1}=I_1+I_2+I_3
\end{aligned}\tag{2.10}$$

For the first term in the right hand side of (2.10), we have

$$\begin{aligned}
I_1=&\sum_{\tilde{m}\in M(m)}(\frac{\partial\theta_m}{\partial\theta_{\tilde{m}}})^2(\frac{S_{\tilde{m}}(S_{\tilde{m}}+1)}{(N_{\tilde{m}}+1)(N_{\tilde{m}}+2)}\\
&-2\theta_{\tilde{m}}\frac{S_{\tilde{m}}}{N_{\tilde{m}}+1}+\theta_{\tilde{m}}^2-(\frac{S_{\tilde{m}}}{N_{\tilde{m}}+1}-\theta_{\tilde{m}})^2)\\
=&\sum_{\tilde{m}\in M(m)}(\frac{\partial\theta_m}{\partial\theta_{\tilde{m}}})^2\frac{S_{\tilde{m}}(1-S_{\tilde{m}}/(N_{\tilde{m}}+1))}{(N_{\tilde{m}}+1)(N_{\tilde{m}}+2)}\\
=&\sum_{\tilde{m}\in M(m)}(\frac{\partial\theta_m}{\partial\theta_{\tilde{m}}})^2\frac{\theta_{\tilde{m}}(1-\theta_{\tilde{m}})}{N_{\tilde{m}}+1}+O_p(N_m^{-\frac{3}{2}}).
\end{aligned}\tag{2.11}$$

For the second term in the right hand side of (2.10), the following inequality holds,

$$\begin{aligned}
&\int_0^1\cdots\int_0^1(\sum_{k,l\in M(n)}\psi_{k,l}(t_k-\theta_k)(t_l-\theta_l))^2\prod_{\tilde{m}\in M}\tilde{p}_{\tilde{m}})dt_{\tilde{m}}\\
&\le c\sum_{k,l\in M(m)}\int_0^1\int_0^1\prod_{i=k,l}(t_i-\theta_i)^2\tilde{p}_i(t_i)dt_i.
\end{aligned}$$

Which shows that $I_2=O_p(N_m^{-2})$. The third term in the right hand side of (2.10) is dominated by

$$c\cdot\prod_{i=k,l,\tilde{m}}(\int_0^1(t_i-\theta_i)^4\tilde{p}_i(t_i)dt_i)^{\frac{1}{4}}=O_p(N_m^{-\frac{3}{2}}).$$

So (2.9) holds. □

From Lemma 2.2 and Lemma 2.3, it is easy to know that

$$\tilde{n}_m + 1 = \frac{\theta_m(1-\theta_m) + o_p(1)}{\sum\limits_{\tilde{m}\in M(m)} (\frac{\partial\theta_m}{\partial\theta_{\tilde{m}}})^2 \frac{\theta_{\tilde{m}}(1-\theta_{\tilde{m}})}{N_{\tilde{m}}+1} + O_p(N_m^{-\frac{3}{2}})} \tag{2.12}$$

Lemma 2.4 *Under the conditions of Theorem 1.1, for every $m \in M$, the following holds*

$$\lim_{N\to\infty} \tilde{n}_m O_p(N_m^{-\frac{3}{2}}) = 0, \tag{2.13}$$

where $N_m = \min(N_{\tilde{m}}, \tilde{m} \in M(m))$ and $N = \min(n_m, m \in M)$.

Proof We prove this proposition by induction.

When $m \in M_0$, from Condition 1.2, we have

$$\lim_{N\to\infty} \frac{\tilde{n}_m}{N_m^{-\frac{3}{2}}} = 0.$$

So (2.13) holds.

Now suppose that $m \in M\backslash M_0$, and that for every $\tilde{m} \in M(m)$, (2.13) holds. Then from (2.12), we have

$$\frac{\tilde{n}_m+1}{N_{\tilde{m}}^{\frac{3}{2}}} = \frac{\theta_m(1-\theta_m) + o_p(1)}{N_{\tilde{m}}^{\frac{3}{2}}(\sum\limits_{\tilde{m}\in M(m)} (\frac{\partial\theta_m}{\partial\theta_{\tilde{m}}})^2 \frac{\theta_{\tilde{m}}(1-\theta_{\tilde{m}})}{N_{\tilde{m}}+1} + O_p(N_m^{-\frac{3}{2}})}.$$

From Condition 1.2 and the induction, it is to know that

$$\lim_{N\to\infty} \tilde{n}_m O_p(N_m^{-\frac{3}{2}}) = 0,$$

i.e., (2.13) holds. □

Lemma 2.5 *Under the conditions of Theorem 1.1, the following holds*

$$N_1(\frac{\partial\theta_1}{\partial\theta^\tau} I^{-1} \frac{\partial\theta_1}{\partial\theta}) = \theta_1(1-\theta_1) + o_p(1), \tag{2.14}$$

where N_1 is given by (1.1)–(1.7) iteratively.

Proof By Lemma 2.2 and Lemma 2.3, the following holds

$$\tilde{n}_m + 1 = \frac{\theta_m(1-\theta_m) + o_p(1)}{\sum\limits_{\tilde{m}\in M(m)} (\frac{\partial\theta_m}{\partial\theta_{\tilde{m}}})^2 \frac{\theta_{\tilde{m}}(1-\theta_{\tilde{m}})}{N_{\tilde{m}}+1} + O_p(N_m^{-\frac{3}{2}})}. \tag{2.15}$$

Let $\{N_m^*, n_m^*, m \in M\}$ be the collection of positive numbers, which is defined by the following expressions recursively

$$\tilde{n}_m^* = \begin{cases} 0, & m \in M_0, \\ \dfrac{\theta_m(1-\theta_m)}{\sum\limits_{\tilde{m}\in M(m)} (\dfrac{\partial\theta_m}{\partial\theta_{\tilde{m}}})^2 \dfrac{\theta_{\tilde{m}}(1-\theta_{\tilde{m}})}{N_{\tilde{m}}^*}}, & m \notin M_0, \end{cases} \tag{2.16}$$

$$N_m^* = n_m + \tilde{n}_m^*. \tag{2.17}$$

In Theorem 2.1 in [3], it was shown that N_1^*, which is defined by (2.16), (2.17) recursively, is given by the following expression

$$N_1^*(\frac{\partial\theta_1}{\partial\theta^\tau} I^{-1} \frac{\partial\theta_1}{\partial\theta}) = \theta_1(1-\theta_1).$$

Now to prove the lemma. It suffies to verify that the following expression

$$\frac{\tilde{n}_m + 1}{\tilde{n}_m^*} \to 1 \quad \text{for} \quad m \notin M_0, \tag{2.18}$$

$$\frac{M_m + 1}{N_m^*} \to 1 \quad \text{for} \quad m \in M. \tag{2.19}$$

We prove these two formulas by induction. First, suppose that $m \in M_0$, According to the definition of N_m^* and N_m (see (2.17) and (1.1)), we obtain

$$\frac{N_m + 1}{N_m^*} = \frac{n_m + \tilde{n}_m + 1}{n_m} \to 1$$

i.e., (2.19) holds for $m \in M_0$.

Let

$$M_1 = \{m \in M, M(m) \subset M_0\}$$

be the collection of the subsystems which is the father of the last generation subsystem. When $m \in M_1$, by (2.16) and (2.17), the following holds

$$\frac{\tilde{n}_m + 1}{\tilde{n}_m^*} = (1 + o_p(1)) \frac{\sum\limits_{\tilde{m}\in M(m)} (\dfrac{\partial\theta_m}{\partial\theta_{\tilde{m}}})^2 \dfrac{\theta_{\tilde{m}}(1-\theta_{\tilde{m}})}{n_{\tilde{m}}}}{\sum\limits_{\tilde{m}\in M(m)} (\dfrac{\partial\theta_m}{\partial\theta_{\tilde{m}}})^2 \dfrac{\theta_{\tilde{m}}(1-\theta_{\tilde{m}})}{n_{\tilde{m}} + \tilde{n}_m + 1} + O_p(N_m^{-\frac{3}{2}})} \xrightarrow{P} 1,$$

which implies that

$$\frac{N_m + 1}{N_m^*} = \frac{n_m + \tilde{n}_m + 1}{n_m} \xrightarrow{P} 1,$$

i.e., (2.18) and (2.19) holds for $m \in M_1$.

Next suppose that $m \in M$, and that, (2.18) and (2.19) hold for all $\tilde{m} \in M(m)$. By Lemma 2.4,

$$\frac{\tilde{n}_m + 1}{\tilde{n}_m^*} = (1 + o_p(1)) \frac{\sum\limits_{\tilde{m} \in M(m)} (\frac{\partial \theta_m}{\partial \theta_{\tilde{m}}})^2 \frac{\theta_{\tilde{m}}(1-\theta_{\tilde{m}})}{N_{\tilde{m}}^*}}{\sum\limits_{\tilde{m} \in M(m)} (\frac{\partial \theta_m}{\partial \theta_{\tilde{m}}})^2 \frac{\theta_{\tilde{m}}(1-\theta_{\tilde{m}})}{N_{\tilde{m}}+1} + O_p(N_m^{-\frac{3}{2}})} \xrightarrow{P} 1,$$

which implies

$$\frac{N_m + 1}{N_m^*} = \frac{n_m + \tilde{n}_m + 1}{n_m} \xrightarrow{P} 1,$$

i.e., (2.18) and (2.19) hold for m. □

Lemma 2.6 *Under the conditions of Theorem 1.1, the following holds*

$$\sqrt{\frac{\tilde{n}_m + 1}{\theta_m(1-\theta_m)}} (\frac{\tilde{s}_m}{\tilde{n}_m + 1} - \theta_m) \xrightarrow{d} \mathcal{N}(0,1), \quad m \notin M_0, \tag{2.20}$$

and

$$\sqrt{\frac{N_m + 1}{\theta_m(1-\theta_m)}} (\frac{S_m}{N_m + 1} - \theta_m) \xrightarrow{d} \mathcal{N}(0,1), \quad m \in M_0, \tag{2.21}$$

where $\tilde{n}_m, \tilde{s}_m, N_m, S_m$ are given by (1.1)–(1.7).

Proof We prove this lemma by induction. When $m \in M_0$, from C.L.T, it is easy to know that (2.21) holds. Now suppose that $m \in M_1$. Then from (1.6) and Taylor expansion, we have

$$\begin{aligned}
\frac{\tilde{s}_m}{\tilde{n}_m + 1} - \theta_m &= m_1(m) - \theta_m \\
&= \sum_{\tilde{m} \in M(m)} \frac{\partial \theta_m}{\partial \theta_{\tilde{m}}} \int_0^1 (t - \theta_{\tilde{m}}) \tilde{p}_{\tilde{m}}(t) dt \\
&\quad + \frac{1}{2} \sum_{k,l \in M(m)} \int_0^1 \psi_{k,l} \prod_{i=k,l} (t_i - \theta_i) \tilde{p}_i(t_i) dt_i \\
&= \sum_{\tilde{m} \in M(m)} \frac{\partial \theta_m}{\partial \theta_{\tilde{m}}} (\frac{S_{\tilde{m}}}{N_{\tilde{m}} + 1} - \theta_{\tilde{m}}) \\
&\quad + \frac{1}{2} \sum_{k,l \in M(m)} \int_0^1 \psi_{k,l} \prod_{i=k,l} (t_i - \theta_i) \tilde{p}_i(t_i) dt_i.
\end{aligned}$$

From Lemma 2.2, 2.3, we know that the second term in the right hand of the above equality is $o_p(\frac{1}{N_m})$. By C.L.T, we have that

$$\sqrt{\frac{\tilde{n}_m + 1}{\theta_m(1-\theta_m)}} (\frac{s_m}{n_m + 1} - \theta_m) \xrightarrow{d} \mathcal{N}(0,1).$$

i.e., (2.20) holds for $m \in M_1$. The indepence of $S_{\tilde{m}}$ and $\tilde{m} \in M(m)$ ensure that (2.21) holds for $m \in M_1$.

Now suppose that $m \in M$, (2.20) and (2.21) hold for every $\tilde{m} \in M(m)$. Similary, we have

$$\sqrt{\frac{\tilde{n}_m + 1}{\theta_m(1-\theta_m)}}\left(\frac{\tilde{s}_m}{\tilde{n}_m + 1} - \theta_m\right) \xrightarrow{d} \mathcal{N}(0,1).$$

Then from (1.6), we have

$$\sqrt{\frac{N_m + 1}{\theta_m(1-\theta_m)}}\left(\frac{S_m}{N_m + 1} - \theta_m\right) = \sqrt{\frac{n_m}{N_m + 1}}\sqrt{\frac{n_m}{\theta_m(1-\theta_m)}}\left(\frac{s_m}{n_m} - \theta_m\right)$$
$$+\sqrt{\frac{\tilde{n}_m + 1}{N_m + 1}}\sqrt{\frac{\tilde{n}_m + 1}{\theta_m(1-\theta_m)}}\left(\frac{\tilde{s}_m}{\tilde{n}_m + 1} - \theta_m\right).$$

From C.L.T,

$$\sqrt{\frac{\tilde{n}_m}{\theta_m(1-\theta_m)}}\left(\frac{s_m}{n_m} - \theta_m\right) \xrightarrow{d} \mathcal{N}(0,1),$$

and s_m and $\tilde{s}_m$ are independent to each other, we have

$$\sqrt{\frac{N_m + 1}{\theta_m(1-\theta_m)}}\left(\frac{S_m}{N_m + 1} - \theta_m\right) \xrightarrow{d} \mathcal{N}(0,1).$$

Proof of Theorem 1.1 Theorem 1.1 follows by letting $m = 1$ in Lemma 2.5 and Lemma 2.6.

Proof of Theorem 1.2 Let $Z_i, i = 1, \cdots, N_1$ be sequence of i.i.d random variables with

$$P(Z_i = 1) = \underline{\theta}_1,$$
$$P(Z_i = 0) = 1 - \underline{\theta}_1,$$

where $P(A)$ is the condition probability of A given N_1, S_1.

From (1.8) we know that

$$P\left(\frac{\sum_{k=1}^{N_1} Z_i - N_1\underline{\theta}_1}{\sqrt{N_1\underline{\theta}_1(1-\underline{\theta}_1)}} \geq \frac{S_1 - N_1\underline{\theta}_1}{\sqrt{N_1\underline{\theta}_1(1-\underline{\theta}_1)}}\right) = \alpha.$$

By Berry-Esseen Theorem, it is easy to show that for almost all sequence of trials, the sequence of condition distribution of

$$\frac{\sum_{k=1}^{N_1} Z_i - N_1\underline{\theta}_1}{\sqrt{N_1\underline{\theta}_1(1-\underline{\theta}_1)}}$$

tends to the standardized normal distribution, therefore

$$-u_\alpha - \frac{S_1 - N_1\theta_1}{\sqrt{N_1\underline{\theta}_1(1-\underline{\theta}_1)}} + \sqrt{\frac{N_1}{\underline{\theta}_1(1-\underline{\theta}_1)}}(\underline{\theta}_1 - \theta_1) \to 0 \quad \text{wp } 1,$$

which shows that the theorem holds. □

References

[1] H.F.Martz and R.A.Waller, *Bayesian Reliability Analysis*, John Wiley & Sons, 1982.

[2] Zheng Zhongguo, *The Virtual System Method for Estimation of Parameter in System* , Tree. Tech. Report, Peking University, 1991.

[3] Zheng Zhongguo, *Virtual System Method and its Application* , Tech. Report, Peking University, 1991.

[4] Zheng Zhongguo and Fang Kaitai, *Large Sample Property of Estimates of Multiparameters in a System Tree* , Tech. Report of Hong Kong Baptist College, Dept. of Math., No.13, 1992.

[5] N.R.Mann et al, *Approximately Optimum confidence bounds for system reliability based on component data* , Techaometrics, **16:3**(1974), 335–347.

[6] Sheng Zhou and Fang Dayin, *The Comparation of the Baysion Confidence Limit of the Reliability of the Binomal Series System Which Prior Distribuction Is B(0,0) with Classical Accurate Solution and the Result of AO Method (Chinese)* , Mathematical Statistics and Applied Probability (Supplymentary Issure), 1987, 161–164.

系统树中的近似 Bayes 方法

郑忠国　戴明[†]　石坚[‡]

(北京大学概率统计系，100871)

摘要

本文利用近似 Bayes 方法对一个具有树形结构的成败型系统的可靠性进行估计. 本文证明了利用近似 Bayes 方法对系统的可靠性参数的估计，其点估计在 Fisher 意义下是渐近有效的，其相应的置信下限也是 Fisher 意义下渐近有效的.

[†]现在工作单位：北京理工大学.

[‡]现在工作单位：中国科学院系统科学研究所.