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On Steffensen's Inequality *

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Theorem 1 *Assume that two integrable functions f and g are defined on the intervel (a,b), that f never increases and that $0 \le g(t) \le 1$ in (a,b). Then steffensen's inequality [1] asserts that $\int_{b-\lambda}^{b} f(t)dt \le \int_a^b f(t)g(t)dt \le \int_a^{a+\lambda} f(t)dt$, where $\lambda = \int_a^b g(t)dt$.*

Denote $\Gamma(f) = \{f(t) | f(t)$ be a arbitrary monotone decreasing function, $t \in [a,b]\}$.

Our main result is

Theroem *Let $g(t)$ be an integrable function in $[a,b]$. Then for $f(t) \in \Gamma(f)$,*

$$\int_a^b f(t)g(t)dt \le \int_a^{a+\lambda} f(t)dt,$$

holds if and only if for any $\xi \in [a,b]$, $\int_a^{\xi} g(t)dt \le \xi - a$, $\int_{\xi}^{b} g(t)dt \ge 0$, where $\lambda = \int_a^b g(t)dt$.

Symmetrically

$$\int_{b-\lambda}^{b} f(t)dt \le \int_a^b f(t)g(t)dt$$

holds if and only if for any $\eta \in [a,b]$, $\int_a^{\eta} g(t)dt \ge 0$, $\int_{\eta}^{b} g(t)dt \le b - \eta$.

It is interesting thing, $\int_a^{\xi} g(t)dt \le \xi - a$ and $\int_{\xi}^{b} g(t)dt \ge 0$ just corresponding to $g(t) \ge 0$ and $g(t) \le 1$. If we modify the conditions in Theorem, we can obtain.

Corollary *Let $g(t)$ be an integrable function in $[a,b]$ such that $g(t) \ge 0$. Then for $f(t) \in \Gamma(f)$, $\int_a^b f(t)g(t)dt \le \int_a^{a+\lambda} f(t)dt$ holds if and only if for any $\zeta \in [a,b]$, $\int_a^{\zeta} g(t)dt \le \zeta - a$, where $\lambda = \int_a^b g(t)dt$.*

Similarly, if transfer $g(t) \ge 0$ to $g(t) \le 1$, the necessary and sufficient condition in Corollary change into $\int_{\zeta}^{b} g(t)dt \ge 0$.

References

[1] D.S.Mitrinović, *Analytic Inequalities* , Springer-Verlag Berlin Heidelberg New York, 1970.
[2] Y.Cao, *The Correction of Extension of Steffensen's Inequality*, Journal of Mathematical Research and Exposition, **11:1** (1991), 151–153.

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