From this and the induction hypothesis, we have $x_2 \in (x_1, \cdots, x_k)^{n-1} : m$. Continue the process we may express

$$y = a_1 x_1 + a_2 x_2 + \cdots + a_k x_k + z,$$

where $z \in (0 : m)$ and $x_i \in (a_1, \cdots, a_k)^{n-1} : m (1 \leq i \leq k)$. Hence $y \in (x_1, \cdots, x_k)^n : m$ and the proof is completed. □

Corollary 2.3 *Let A and $a_1, \cdots, a_d$ be as in Theorem 2.1 then* ·

$$(a_1, \cdots, a_k)^n \cap (a_{k+1}, \cdots, a_d)^m \subseteq (a_1, \cdots, a_k)^{n-1} (a_{k+1}, \cdots, a_d)^m,$$

and

$$(a_1, \cdots, a_k)^n \cap (a_{k+1}, \cdots, a_d)^m \subseteq (a_1, \cdots, a_k)^n (a_{k+1}, \cdots, a_d)^{m-1}$$

for all integers n, m and $1 \leq i < d$.

This follows immediately from Theorem 2.1 and Lemma 2.2.

Theorem 2.4 *Let A be a quasi-Buchsbaum ring of dimension d, and $a_1, a_2, \cdots, a_d$ a system of parameters for A contained in m^2. Then $A/(a_1, a_2, \cdots, a_k)^n$ is quasi-Buchsbaum for any positive integer n and $1 \leq k < d$.*

The proof is similar to that of Theorem 2.1 and is omitted.

References

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关于 Buchsbaum 环的注记

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摘 要

本文证明了下述结论. 设 A 是一个级数为 d 的 Buchsbaum 环，$(a_1, a_2, \cdots, a_n)$ 是 A 的一个参数系统，则任何正整数 $n, A/(a_1, a_2, \cdots, a_k)^n (1 \leq k \leq d)$ 仍是 $d-k$ 维的 Buchsbaum 环.

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A Note on Buchsbaum Rings *

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Abstract In this paper we obtain the following result: Let A be a Buchsbaum ring of $\dim A = d$ and $(a_1, a_2, \cdots, a_d)$ be a parameter of system for A, then $A/(a_1, \cdots, a_k)^n$ is also a Buchsbaum ring of dimension $(d-k)$, where $1 \le k \le d$ and n is a positive integer.

Keywords Buchsbaum ring, weak m-sequence.

Classification AMS(1991)13A30/CCL O153.3

1 Introduction

Let A be a Noetherian local ring with maximal ideal m and $\dim A = d$. We say that A is Buchsbaum (resp. quasi-Buchsbaum) if every system $a_1, a_2, \cdots, a_d$ of parameters for A (resp. at least one, and hence every, system $a_1, a_2, \cdots, a_d$ of parameters for A contained in m^2) form a weak m-sequence, that is, the equality $(a_1, \cdots, a_{i-1}) : a_i = (a_1, \cdots, a_{i-1}) : m$ holds for any $1 \le i \le d$, where $a_{-1} = 0$ cf.[3](resp.[4]). It is well known that every system $a_1, \cdots, a_d$ of parameters for a Buchsbaum ring A is a d-sequence, i.e., the equality $(a_1, \cdots, a_{i-1}) : a_i a_j = (a_1, \cdots, a_{i-1}) : a_i$ holds for all $1 \le i \le d$ and $i \le j \le d$.

2 Main Result

Let A be a Cohen-Macaulay Noetherian local ring of dimension d and let $a_1, \cdots, a_d$ be a system of parameters. It is known that $A/(a_1, \cdots, a_i)^n$ is also a Cohen-Macaulay ring of dimension $(d-i), (1 \le i < d)$[2]. What happens for A to be a Buchsbaum ring? We obtain the following Theorem 2.1 which gives a positive answer to the question.

Theorem 2.1 *Let A be a Buchsbaum local ring of dimention d and let $a_1, \cdots, a_d$ be a system of parameters for A. Then $A/(a_1, \cdots, a_k)^n$ is a Buchsbaum ring for $1 \le k < d$ and $n \ge 1$.*

We quote a result from [1] which will be used several times in the proof of theorem 2.1.

Lemma 2.2 *Let A be a commutative ring, $a_1, \cdots, a_n$ a d-sequence modulo an ideal I of A, and $X = (a_1, \cdots, a_n)$, then*

$$X^m \cap I \subseteq X^{m-1} I$$

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for all $m \geq 1$.

Proof of Theorem 2.1 We put $A/(a_1, \cdots, a_k)^n = \overline{A}$ and denote by $m, \overline{m}$ the maximal ideal of $A, \overline{A}$ respectively. Let $\pi : A \to \overline{A}$ be the natural projection and denote $\pi(a) = \overline{a}$ for $a \in A$. If $\overline{a}_{k+1}, \cdots, \overline{a}_d$ is a system of parameters for $\overline{A}$, then $a_1, \cdots, a_k, a_{k+1}, \cdots, a_d$ is a system of parameters for A. Clearly

$$(\overline{a}_{k+1}, \cdots, \overline{a}_{i-1}) : \overline{a}_i = ((a_1, \cdots, a_k)^n, a_{k+1}, \cdots, a_{i-1}) : a_i$$

and

$$(\overline{a}_{k+1}, \cdots, a_{i-1}) : \overline{m} = ((a_1, \cdots, a_k)^n, a_{k+1}, \cdots, a_{i-1}) : m,$$

where $i \geq k+1$. We know that $A/a_{k+1}A, \cdots, A/(a_{k+1}, \cdots, a_{d-1})A$ are Buchsbaum rings. Hence, in order to prove that $\overline{a}_{k+1}, \cdots, \overline{a}_d$ is a weak $\overline{A}$-sequence, it is sufficient to prove that $(a_1, \cdots, a_k)^n : a_{k+1} = (a_1, \cdots, a_k)^n : m$.

For $k = 1$ or $n = 1$, the conclusion follows immediately from the properties of Buchsbaum rings. Hence we can assume $i \geq 2$ and $n \geq 2$. By induction on n, suppose the conclusion holds for all positive integers less than n. Let $y \in (a_1, \cdots, a_k)^n : a_{k+1}$, i.e., $a_{k+1}y \in (a_1, \cdots, a_k)^n$. By Lemma 2.2, we have $a_{k+1}y \in a_{k+1}(a_1, \cdots, a_k)^{n-1}$. Thus $y \in (a_1, \cdots, a_k)^{n-1} + (0 : m)$. Since

$$(a_1, \cdots, a_k)^{n-1} = a_1(a_1, \cdots, a_k)^{n-2} + (a_2, \cdots, a_k)^{n-1}, \tag{1}$$

we can write $y = a_1x_1 + x_1' + z$, where $x_1 \in (a_1, \cdots, a_k)^{n-2}$ and $x_1' \in (a_2, \cdots, a_k)^{n-1}$. So

$$a_{k+1}a_1x_1 + a_{k+1}x_1' \in (a_1, \cdots, a_k)^n.$$

As in (1), we have

$$a_{k+1}a_1x_1 + a_{k+1}x_1' = a_1y_1 + y_1',$$

where $y_1 \in (a_1, \cdots, a_k)^{n-1}$ and $y_1' \in (a_2, \cdots, a_k)^n$. Hence

$$a_1(a_{k+1}x_1 - y_1) \in (a_2, \cdots, a_k)^{n-1}.$$

By the induction hypothesis, we have that

$$a_{k+1}x_1 - y_1 \in (a_2, \cdots, a_k)^{n-1} : m.$$

Therefore $a_{k+1}^2x_1 \in (a_1, \cdots, a_k)^{n-1}$. By the induction hypothesis again, we have $x_1 \in (a_1, \cdots, a_k)^{n-1} : m$.

Now, $a_{k+1}x_1' \in (a_1, \cdots, a_k)^n$ and $x_1' \in (a_2, \cdots, a_k)^{n-1}$. Aslo as in (1), we can write $x_1' = a_2x_2 + x_2'$, where $x_2 \in (a_2, \cdots, a_k)^{n-2}$ and $x_2' \in (a_3, \cdots, a_k)^{n-1}$. We use $(a_1, \cdots, \hat{a}_i, \cdots, a_k)$ to denote $(a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_k)$, and express

$$a_{k+1}a_2x_2 + a_{k+1}x_2' = a_2y_2 + y_2',$$

where $y_2 \in (a_1, \cdots, a_k)^{n-1}$ and $y_2' \in (a_1, \hat{a}_2, a_3, \cdots, a_k)^n$. So

$$a_2(a_{k+1}x_2 - y_2) \in (a_1, \hat{a}_2, a_3, \cdots, a_k)^{n-1}.$$

From this and the induction hypothesis, we have $x_2 \in (x_1, \cdots, x_k)^{n-1} : m$. Continue the process we may express

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