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代数体函数的唯一性问题

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摘要

讨论了重值和 Valiron 亏量对代数体函数唯一性问题的影响，证明了两个唯一性定理.

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The Unique Problem of Algebroidal Function *

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Abstract This paper discusses the influence which multiple value and valiron deficiencies exercice upon the unique problem of algebroid function. We establish two unique theorems, which extend some previous results.

Keywords algebroid function, multiple value and deficiencies, unique problem.

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1. Introduction

In 1980, H.Ueda discussed the influence which multiple value and Nevalinna deficiencies or Valiron deficiencies exercice upon the unique problem of meromorphic function. On the unique problem of algbroid function, it was considered first by Valiron[2], afterwards, He Yuzan has studied it in a systematic way[3],[4],[5]. In 1985, he obtained an unique theorem of algebroid function with multiple value and Nevenlinna deficiencies[5]. But how do Valiron deficinecies influence the uniqueness of algebroid function? This paper considers the problem and proves the following result

Theorem 1 *Let $w(z)$ and $\tilde{w}(z)$ be v-value and u-value algebroid function with finite order, respectively, detenmined by*

$$\psi(z,w) \equiv A_v(z)w^v + A_{v-1}(z)w^{v-1} + \cdots + A_0(z) = 0,$$
$$\phi(z,\tilde{w}) \equiv B_u(z)\tilde{w}^u + B_{u-1}(z)\tilde{w}^{u-1} + \cdots + B_0(z) = 0,$$

where $u \le v$, all $A_i(z)(i = 0,1,\cdots,v)$ be holomorphic, which have no common zeros, so do all $B_j(z)(j = 0,1,\cdots,u)$. Suppose that $a_j(j = 1,2,\cdots,p)$ be p distinct complex values, $k_j(j = 1,2,\cdots,p)$ be p positive integers, $\overline{E}_{k_j}(a_j,w)$ and $\overline{E}_{k_j}(a_j,\tilde{w})$ be the sets of a_j value points of $w(z)$ and $\tilde{w}(z)$, separately, and a_j value point is with multiple$\le k_j$, every a_j value point only count once, and that

$$\overline{E}_{k_j}(a_j,w) = \overline{E}_{k_j}(a_j,\tilde{w}), \quad j = 1,2,\cdots,p. \tag{1}$$

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If

$$\sum_{j=1}^{p}\frac{k_j}{k_j+1}-2v\frac{2k+1}{k+1}=0,\quad k=\max_{1\le j\le P}k_j, \tag{2}$$

$$\frac{1}{k+1}\sum_{j=1}^{P}\max\{\triangle(a_j,w),\triangle(a_j,\tilde{w})\}>0,$$

then $w(z)\equiv\tilde{w}(z)$.

Define

$$\triangle(a,w,\tilde{w})=1-\varliminf_{r\to\infty}\frac{N(r,\frac{1}{w-a})+N(r,\frac{1}{\tilde{w}-a})}{T(r,w)+T(r,\tilde{w})},$$

$$\delta(a,w,\tilde{w})=1-\varlimsup_{r\to\infty}\frac{N(r,\frac{1}{w-a})+N(r,\frac{1}{\tilde{w}-a})}{T(r,w)+T(r,\tilde{w})}.$$

Theorem 2 *Under the hypotheses of Theorem 1, if*

$$\sum_{j=1}^{P}\frac{k_j}{k_j+1}-2v\frac{2k+1}{k+1}\le 0,$$

$$\max_{1\le m\le p}\frac{1}{k+1}\{\triangle(a_m,w,\tilde{w})+\sum_{\substack{j=1\\ j\ne m}}^{p}\delta(a_m,w,\tilde{w})\}>2v\frac{2k+1}{k+1}-\sum_{j=1}^{P}\frac{k_j}{k_j+1}, \tag{3}$$

then $w(z)\equiv\tilde{w}(z)$.

Esecially, we have

Corollary *Under the hypotheses of Theorem 1, if*

$$\sum_{j=1}^{P}\frac{k_j}{k_j+1}-2v\frac{2k+1}{k+1}=0,$$

$$\max_{1\le j\le p}\triangle(a_j,w,\tilde{w})>0,$$

then $w(z)\equiv\tilde{w}(z)$.

2. Lemma

According to Theorem 2.22[6,p.97], for algebroid function $w(z)$ with finite order, we have

$$(p-2v)T(r,w)<\sum_{j=1}^{p}\overline{N}(r,\frac{1}{w-a_j})+O(\log r).$$

Again

$$\overline{N}(r,\frac{1}{w-a_j})\le\frac{k_j}{k_j+1}\overline{N}_{k_j}(r,\frac{1}{w-a_j})+\frac{1}{k_j+1}N(r,\frac{1}{w-a_j})\quad(j=1,2,\cdots,p),$$

we get

Lemma 1 *For algebroid function $w(z)$ with finite order, we have*

$$(p-2v)T(r,w) < \sum_{j=1}^{p}\frac{k_j}{k_j+1}\overline{N}_{k_j}(r,\frac{1}{w-a_j}) + \sum_{j=1}^{p}\frac{1}{k_j+1}N(r,\frac{1}{w-a_j}) + O(\log r). \quad (4)$$

Lemma 2 *For algebroid function $w(z)$ and $\tilde{w}(z)$ with finite order, we have*

$$\varliminf_{r\to\infty}\frac{T(r,w)}{T(r,\tilde{w})} > 0.$$

Proof Let $\bar{n}_k^0(r,a)$ indicate the number of the common value point of $w(z)=a$ and $\tilde{w}(z)=a$ in $|z|<r$, and the multiple of every value point not greater than k, every value point count only once, suppose that

$$\overline{N}_k^0(r,a) = \frac{u+v}{2uv}\int_0^r \frac{\bar{n}_k^0(t,a)-\bar{n}_k^0(0,a)}{t}dt + \frac{u+v}{2uv}\bar{n}_k^0(0,a)\log r,$$

$$\overline{N}_k^{12}(r,a) = \overline{N}_k(r,\frac{1}{w-a}) + \overline{N}_k(r,\frac{1}{\tilde{w}-a}) - 2\overline{N}_k^0(r,a). \quad (5)$$

If $w(z)$ and $\tilde{w}(z)$ satisfy (1), then

$$\overline{N}_k^{12}(r,a_j) = 0,\quad j=1,2,\cdots,p. \quad (6)$$

We deduce from (1), (5), (6) that

$$\overline{N}_{k_j}(r,\frac{1}{\tilde{w}-a_j}) \le \frac{u+v}{u}\overline{N}_{k_j}(r,\frac{1}{w-a_j}),\quad j=1,2,\cdots,p.$$

If the Lemma is not true, then there exists $\{r_n\}\nearrow\infty$ such that $\lim_{n\to\infty}\frac{T(r_n,w)}{T(r_n,\tilde{w})}=0$. By Lemma 1, we get

$$(p-2u)T(r_n,\tilde{w}) < \sum_{j=1}^{p}\frac{k_j}{k_j+1}\overline{N}_{k_j}(r_n,\frac{1}{\tilde{w}-a_j}) + \sum_{j=1}^{p}\frac{1}{k_j+1}N(r_n,\frac{1}{\tilde{w}-a_j}) + O(\log r). \quad (7)$$

Notice that $k=\max_{1\le j\le p}k_j$, simplify the above sequal, we have

$$(\sum_{j=1}^{p}\frac{k_j}{k_j+1} - 2u)T(r_n,\tilde{w}) < \frac{u+v}{u(k+1)}T(r_n,w) + O(\log r)$$

This implies that

$$\sum_{j=1}^{p}\frac{k_j}{k_j+1} - 2u \le 0$$

which is impossible, because of (2) and $u\le v$. So Lemma 2 is true.

Lemma 3 *If $\triangle(a,w)>0$, then $\triangle(a,w,\tilde{w})>0$.*

Proof Since

$$\triangle(a,w)=1-\varliminf_{r\to\infty}\frac{N(r,\frac{1}{w-a})}{T(r,w)}>0,$$

thus

$$c=1-\triangle(a,w)=\varlimsup_{r\to\infty}\frac{N(r,\frac{1}{w-a})}{T(r,w)}<1.$$

Take $\varepsilon>0$ such that $c+\varepsilon=c'<1$, then there exists $\{r_n\}\nearrow\infty$ such that

$$\frac{N(r_n,\frac{1}{w-a})}{T(r_n,w)}<c'<1.$$

Notice the basic fact $N(r_n,\frac{1}{\tilde{w}-a})\le T(r_n,\tilde{w})$, we have

$$\frac{N(r_n,\frac{1}{w-a})+N(r_n,\frac{1}{\tilde{w}-a})}{T(r_n,w)+T(r_n,\tilde{w})}<c'+(1-c')\frac{1}{1+\frac{T(r,w)}{T(r_n,\tilde{w})}}.$$

We deduce from the quality of upper limit and lower limit and Lemma 2 that

$$\varlimsup_{n\to\infty}\frac{N(r_n,\frac{1}{w-a})+N(r_n,\frac{1}{\tilde{w}-a})}{T(r_n,w)+T(r_n,\tilde{w})}<1.$$

This imply that $\triangle(a,w,\tilde{w})>0$. This completes that proof of Lemma 3.

3. The proofs of Theorems

The proof of Theorem 2.

For $\tilde{w}(z)$, according to Lemma 1, we have

$$(p-2u)T(r,\tilde{w})<\sum_{j=1}^{p}\frac{k_j}{k_j+1}\overline{N}_{k_j}(r,\frac{1}{\tilde{w}-a_j})+\sum_{j=1}^{p}\frac{1}{k_j+1}N(r,\frac{1}{\tilde{w}-a_j})+O(\log r).$$

Combining (4), keeping in mind that $u\le v$ and $k=\max_{1\le j\le p}k_j$, we may verify that

$$(p-2v)(T(r,w)+T(r,\tilde{w}))<\frac{k}{k+1}\sum_{j=1}^{p}\{\overline{N}_{k_j}(r,\frac{1}{w-a_j})+\overline{N}_{k_j}(r,\frac{1}{\tilde{w}-a_j})\}$$

$$+\sum_{j=1}^{p}\frac{1}{k_j+1}\{N(r,\frac{1}{w-a_j})+N(r,\frac{1}{\tilde{w}-a_j})\}+O(\log r). \qquad (8)$$

If $w(z)z\not\equiv\tilde{w}(z)$, then

$$\sum\overline{N}^0(r,a)\le n(r,\frac{1}{R(\psi,\Phi)}),$$

where $R(\psi,\Phi)$ is the nodal expression of $\psi(z,w)$ and $\Phi(z,\tilde{w})$, that is to say

$$R(\psi,\Phi)=[A_v(z)]^u[B_u(z)]^v\prod_{\substack{1\leq j\leq v\\ 1\leq l\leq u}}[w_j(z)-\tilde{w}_l(z)].$$

By Jensen Formula, we get

$$\begin{aligned} N(r,\frac{1}{R(\psi,\Phi)}) &= \frac{1}{2\pi}\int_0^{2\pi}\log|R(\psi,\Phi)|d\theta+\log|\frac{1}{R(\psi,\Phi)}|_{z=0}\\ &= \frac{u}{2\pi}\int_0^{2\pi}\log|A_v(r\exp(i\theta)|d\theta+\frac{v}{2\pi}\int_0^{2\pi}\log|B_u(r\exp(i\theta)|d\theta\\ &\quad +\frac{1}{2\pi}|\prod_{\substack{1\leq j\leq v\\ 1\leq l\leq u}}[w_j(r\exp(i\theta)-\tilde{w}_l(r\exp(i\theta))]|d\theta+O(1)\\ &\leq uv(T(r,w)+T(r,\tilde{w}))+O(1). \end{aligned}$$

Hence

$$\sum\overline{N}^0(r,a)\leq\frac{2uv}{u+v}(T(r,w)+T(r,\tilde{w}))+O(1)\leq v(T(r,w)+T(r,\tilde{w}))+O(1). \qquad (9)$$

Notice that $w(z)$ and $\tilde{w}(z)$ satisfy (1), from (5), (6) and (9), we can simply (8) and get

$$(p-2v\frac{2k+1}{k+1})(T(r,w)+T(r,\tilde{w}))<\sum_{j=1}^{p}[N(r,\frac{1}{w-a_j})+N(r,\frac{1}{\tilde{w}-a_j})]+O(\log r).$$

Go on simplification, we have

$$\frac{1}{k+1}\sum_{j=1}^{p}[1-\frac{N(r,\frac{1}{w-a_j})+N(r,\frac{1}{\tilde{w}-a_j})}{T(r,w)+T(r,\tilde{w})}]<O(1)+2v\frac{2k+1}{k+1}-\sum_{j=1}^{p}\frac{k_j}{k_j+1}. \qquad (10)$$

Take upper limit in above sequel, we obtain

$$\frac{1}{k+1}\max_{1\leq j\leq p}\{\triangle(a_m,w,\tilde{w})+\sum_{\substack{j=1\\ j\neq m}}^{p}\delta(a_j,w,\tilde{w})\}\leq 2v\frac{2k+1}{k+1}-\sum_{j=1}^{p}\frac{k_j}{k_j+1}$$

which contradicts to (3).

This completes the proof of theorem 2.

The Corollary is obvious, and Theorem 1 follows from Lemma 3 and Corollary.

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