Hence we have

$$f(g) + \rho f(x - g_0) \leq f[(x - g_0) - g],$$

that is, $0 \in R^{\rho}_{f,Y}(x - g_0)$. Since f is a sublinear function, we have $0 \in R^{\rho}_{f,Y}[t(x_0 - g_0)]$ for each $t \geq 0$. This implies that, for every $t > 0$,

$$f(-g) + t\rho f(x - g_0) \leq f[t(x - g_0) + g]. \tag{3.1}$$

Since f is symmetric, (3.1) implies that

$$\rho f(x - g_0) \leq [f[t(x - g_0) + g] - f(g)/t.$$

Hence $\tau_f(g, x_0 - g_0) \geq \rho f(x_0 - g_0)$. □

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局部凸空间的余逼近

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摘 要

研究了在局部凸空间中的 f- 余逼近和强 f- 余逼近的一些性质.

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Coapproximation in Locally Convex Spaces *

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Abstract The characterizations of f-coapproximation and strongly f-coapproximation in locally convex spaces are given.

Keywords f-coapproximation, strongly f-coapproximation, f- proximinal.

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1. Introduction

Let X and X' be a pair of linear spaces put in duality by a bilinear form $<\ ,\ >$. We assume that this bilinear form $<\ ,\ >$ is separating, i.e., for each $x \in X$ and $x \neq 0$, there exists a y in X' such that $< x, y > \neq 0$ and, for each $y \in X'$ and $y \neq 0$, there exists an $x \in X$ such that $< x, y > \neq 0$. A topology on X is said to be *compatible* if it is a separated locally convex topology for which continuous linear functions on X are precisely of the form

$$< \cdot, y >: x \to < x, y >, \quad \text{for} \quad y \in X'.$$

Let f be a continuous convex function defined on X and satisfy $f(0) = 0$. Given a nonempty Y of X and $x \in X$, let

$$f_Y(x) = \inf\{f(x - y); \quad y \in Y\};$$

$$P_{f,Y}(x) = \{y \in Y; \quad f_Y(x) = f(x - y)\};$$

The set-valued mapping $P_{f,Y}$ is called f-metric projection from X onto Y. Y is said to be f-proximinal (resp. f-Chebyshev) if $P_{f,Y}(x)$ is nonempty (resp. $P_{f,Y}(x)$ is a singleton) for each $x \in X$.

Let X be a locally convex space, Y a nonempty subset of X and f a real-valued function defined on X. For $x \in X$ and $g_0 \in Y$, if for every $g \in Y$,

$$f(g - g_0) \leq f(x - g),$$

then g_0 is called to be best f-coapproximation of x with respect to Y. We denote by $R_{f,Y}(x)$ all the best f-coapproximation of x with respect to Y.

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In section 2, we study the properties of f-coapproximations.

Given $0 < \rho < 1$, if for every $g \in Y, f(g - g_0) + \rho f(x - g_0) \leq f(x - g)$, then g_0 is called to be strongly best f-coapproximation of x with respect to Y. We denote by $R^{\rho}_{f,Y}(x)$ all the strongly best f-coapproximation of x with respect to Y.

In section 3,we study the properties of strongly f-coapproximation. For $r > 0$, let

$$S_r = \{x \in X;\ \ f(x) \leq r\}$$

denote the sub-level subset of f, and $P_r(x) = \inf\{\lambda > 0; x \in \lambda S_r\}$ denote the Minkowski gauge of S_r. Then P_r is a non-negative continuous sublinear function.

In this paper, we consider the following conditions:

(F1) There exists a continuous bijection $\psi : R_+ \mapsto R_+$ such that, for any $x \in X$ and $\lambda \geq 0$, $f(\lambda x) = \psi(\lambda) f(x)$ and f is continuous and convex.

(F2) f is a symmetric and sublinear function.

Obviously, if f satisfies the condition (F2), then f satisfies the condition (F1), and if there exists an $x \in X \backslash \{0\}$ such that $f(x) > 0$, then ψ is a convex function and $\psi(t) \to \infty$ as $t \to \infty$.

Lemma P.G. (P.Govindarajulu and D.V.Pai [1]) *Suppose f satisfies the condition (F1) and $0 = f(0) \leq f(x)$. Then for any $\alpha, \beta > 0$, $S_\alpha = (1/\beta) S_{\psi(\beta)\alpha}$, $P_\alpha = \beta P_{\psi(\beta)\alpha}$.*

By this Lemma, we have $P_{P_\alpha, Y}(x) = P_{P_\beta, Y}(x)$, for any $x \in X$ and $\alpha, \beta > 0$.

2. The Best f-coapproximation

Let f be a real-valued continuous convex function defined on X. Given $x, y \in X$, obviously, $h(t) = f(x + ty)$ is a convex function. Thus the limit

$$\tau_f(x\ ,\ y) = lim_{t \mapsto 0+}[h(t) - h(0)]/t$$

is well defined.

The following Lemma is trivial.

Lemma 2.1. *Let X and f be as above. Then $\tau_f(x, y) \leq 0$ if and only if, for each $t > 0$, $f(x + ty) \leq f(x)$.*

Let

$$\begin{aligned} R'_{f,Y} &= \{g_0 \in Y;\ \text{for every}\ \ g \in Y, \tau_f(g_0 - g, x - g_0) \geq 0\}. \\ A_{f,Y} &= \{u \in X;\ \text{for every}\ \ g \in Y, f(u - g) \leq f(x - g)\}. \\ A'_{f,Y} &= \{u \in Y;\ \text{for every}\ \ g \in Y, \tau_f(u - g, x - u) \geq 0\}. \end{aligned}$$

Obviously,

$$R'_{f,Y} \subseteq R_{f,Y};\ \ A'_{f,Y} \subseteq A_{f,Y};\ \ R'_{f,Y} = R_{f,Y} \cap Y;\ \ A'_{f,Y} = A_{f,Y} \cap Y.$$

Lemma 2.2. *Let f satisfy the continuous convex function defined on X, Y a subspace of X and $x \in X \setminus Y$. Then $u \in A'_{f,Y}(x)$ if and only if, for every $g \in Y$, $0 \in P_{f,R_-[u-x]}(u - g)$*

where $R_-[u-x] = \{t(u-x); t \leq 0\}$.

Proof By lemma 2.1, if $u \in A'_{f,Y}(x)$, then, for every $t \leq 0$, we have

$$f[(u-g) - tf(x-u)] \geq f(u-g).$$

So $0 \in P_{f,R_-[u-x]}(u-g)$.

The inverse proof of the other part is similar. □

Theorem 2.3 *Let X be a locally convex space and f a real-valued function defined on X which satisfies the condition (F2). Let Y be subset of X, $x_0 \in X \setminus Y$ and $g_0 \in Y$. Then $g_0 \in R'_{f,Y}(x_0)$ if and only if, for every $g \in Y$, when $f(g-g_0) \neq 0$, there exists a $\phi \in X'$ such that*

(a) For every $u \in X$, $\phi(u) \leq f(u)$.

(b) $\phi(x_0 - g_0) \geq 0$.

(c) $\phi(g_0 - g) = f(g_0 - g)$.

Proof By (b), we have $\phi[t(g_0 - x_0)] \leq 0$ for $t \geq 0$. By (a), we have

$$\begin{aligned} f(g_0 - g) &= \phi(g_0 - g) \leq \phi(g_0 - g) - \phi[t(g_0 - x_0)] \\ &= \phi[(g_0 - g) - t(g_0 - x_0)] \leq f[(g_0 - g) - t(g_0 - x_0)]. \end{aligned}$$

Hence $\tau_f(g_0 - g, x_0 - g_0) \geq 0$ and $g_0 \in R'_{f,Y}(x_0)$. we complete the proof of sufficience.

Let $g_0 \in R'_{f,Y}(x_0)$. Given $g \in Y$, assume that $f(g - g_0) = r > 0$. Since $\tau_f(g_0 - g, x_0 - g_0) \geq 0$, by lemma 2.1, for each $t \geq 0$,

$$f(g_0 - g) \leq [(g_0 - g) + t(x_0 - g_0)] = f[(g_0 - g) - t(g_0 - x_0)]. \tag{2.1}$$

Hence $0 \in P_{f,R_+[g_0 - x_0]}(g_0 - g)$. Let

$$B = \{v \in X;\ \ f[v - (g_0 - g)] < r\}.$$

Then B is nonempty and an open subset of X since f is continuous. By (2.1), $B \cap R_+[g_0 - x_0] = \emptyset$. By separable Theorem, There exist $\phi_0 \in X'$ and $c_0 \in R$ such that, when $y \in B$ and $t \geq 0$,

$$\phi_0[t(x_0 - g_0)] \leq c_0 < \phi(y). \tag{2.2}$$

Obviously, $c_0 \geq 0$. Hence we have

$$\phi_0(g_0 - x_0) \leq c_0/t \mapsto 0,\ (t \mapsto \infty)$$

that is, $\phi_0(x_0 - g_0) \geq 0$. Since $g_0 - g \in B$, by (2.2), we have $\phi_0(g_0 - g) > 0$. Let $\phi = r \cdot \phi_0/\phi_0(g_0 - g)$. Evidently, (b) and (c) hold. It remains to prove (a). Let $V = \{v \in X;\ \ f(v) < r\}$. For any $v \in V$, $v + g_0 - g \in B$. So $\phi(g_0 - g) \geq \phi(-v)$. Since f is symmetric, so V is symmtric. Hence, for every $v \in V$, $\phi(v) \leq \phi(g_0 - g) = r$. Since f and ϕ are continuous, we have, when $f(v) = r$, $\phi(v) \leq r$. Since f is sublinear, so (a) holds.

Theprem 2.4 *Let X, Y and f be as in Theorem 2.3. Then the following statements are equivalent.*

(1) $R'_{f,Y}(x_0) = R_{f,Y}(x_0)$.
(2) If $g_0 \in R_{f,Y}(x_0)$, *then for every* $g \in Y$, $\tau_f(g_0 - g\ ,\ x_0 - g_0) \geq 0$.
(3) If $g_0 \in R_{f,Y}(x_0)$, *then for every* $g \in Y$,

$$\sup\{\phi(x_0 - g_0 0;\ \ \phi \in L_{(g_0 - G)}\} \leq 0$$

where $L_u = \{\phi \in X'$; *for every* $v \in X, \phi(v) \leq f(v)$ *and* $\phi(u) = f(u)\}$.

Proof By definition and Theorem 2.3, (1)⇒(2) and (2) ⇒(3) are trivial.

It remains to show (3) ⇒ (1). Given $g \in Y$, by assumption, for every $\varepsilon > 0$, there exists $\phi \in L_{g_0 - g}$ such that

$$\phi(x_0 - g_0) \geq -\varepsilon.$$

Hence, when $0 < t \leq 1$, we have

$$0 \leq \phi[t(x_0 - g_0] + t\varepsilon.$$

Since $\phi \in L_{g_0 - g}$, we have

$$\begin{aligned} f(g_0 - g) &= \phi(g_0 - g) \leq g(g_0 - g) + t\phi(x_0 - g_0) + t\varepsilon \\ &= \phi[(g_0 - g0 + t(x_0 - g_0)] + t\varepsilon. \end{aligned}$$

Thus $\tau_f(g_0 - g\ ,\ x_0 - g_0) \geq -\varepsilon$. Hence $\tau_f(g_0 - g\ ,\ x_0 - g_0) \geq 0$, that is, $g_0 \in R'_{f,Y}(x_0)$. □

Theorem 2.5 *Let* X, Y *and* f *be as in Theorem 2.3. If* $g_0 \in A'_{f,Y}(x_0)$, *then, when* $t \geq 0$, $g_0 \in A'_{f,Y}[g_0 + t(x_0 - g_0)]$.

Proof Let $v_t = g_0 + t(x_0 - g_0)$. Since $v_0 = g_0$ and ,for every $u \in X$, $\tau_f(u, 0) = 0$, $g_0 \in A'_{f,Y}(v_0)$. Let $t_0 > 0$. Since $g_0 \in A'_{f,Y}(x_0)$, by Lemma 2.1, we have

$$\tau_f(g_0 - g\ ,\ x_0 - g_0) \geq 0.$$

By Lemma 2.1, for every $\lambda > 0$,

$$f[(g_0 - g) + \lambda(x_0 - g_0)] - f(g_0 - g) \leq 0.$$

Hence

$$f[(g_0 - g) + tt_0(x_0 - g_0)] \geq f(g_0 - g),$$

and $\tau_f(g_0 - g, t_0(x_0 - g_0)) \geq 0$. Since $t_0(x_0 - g_0) = v_{t_0} - g_0$, so $g_0 \in A'_{f,Y}(v_{t_0})$. □

Theorem 2.6 *Let* X, Y *and* f *satisfy the conditions in Theorem 2.3. Then the following statements are equivalent.*
(1) For each $x \in X$, $R_{f,Y}(x) = R'_{f,Y}(x)$.
(2) For each $x \in X$ *and* $g_0 \in R_{f,Y}(x)$, *then, for every* $t > 0$,

$$g_0 \in R_{f,Y}[g_0 + t(x - g_0)].$$

(3) For every $g_0 \in Y$, $[R_{f,Y}]^{-1}(g_0)$ *is a cone with the vertix* 0.

Proof (2) $\Leftrightarrow$ (3) is obvious.

In an analogous way to the proof of Theorem 2.5, we can get (1) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1). For every $g \in Y$ and $t > 0$, since $g_0 \in R_{f,Y}[g_0 + t(x - g_0)]$, we have

$$f(g_0 - g) \le f[g_0 + t(x - g_0) - g] = f[(g_0 - g) + t(x - g_0)].$$

Hence $\tau_f(g_0 - g\ ,\ x - g_0) \ge 0$. □

Lemma 2.7 *Let X be locally convex space and f a symmetric real-valued function defined on X which satisfies the condition $(F1)$ and $f(0) = 0$. Then for any $x, y \in X$ and $r > 0$, $\tau_f(x, y) \ge 0$ if and only if $\tau_{P_r}(x, y) \ge 0$.*

Proof By Lemma P.G., given $u, v \in X$, if there exists $r > 0$ such that $P_r(u) \le P_r(v)$, then, for every $\lambda > 0$,

$$P_\lambda(u) \le P_\lambda(v). \tag{2.1}$$

By Lemma 2.1, $\tau_f(x, y) \ge 0$ if and only if

$$f(x) \le f(x + ty). \tag{2.2}$$

By Lemma 2.3 of [6], (2.2) $\Leftrightarrow$, for any $r > 0andt¿0$

$$P_r(x) \ge P_r(x + ty). \tag{2.3}$$

Obviously, (2.3) $\Leftrightarrow \tau_{P_r}(x, y) \ge 0$. □

Remark By Lemma 2.7, the condition (F2) in Theorem 2.3, 2.4 2.5 and 2.6 can be replaced by the conditions in Lemma 2.7.

3. The Strongly f-coapproximation

Lemma 3.1 *Let f be a real-valued function defined on a locally convex space X which is a non-negetive and convex function and $f(0) = 0$ and Y a subset of X. For $x_0 \in X$ and $g_0 \in Y$, if, for every $g \in Y$, $\tau_f(g_0 - g, x_0 - g_0) \ge \rho f(x_0 - g_0)$, then $g_0 \in R^\rho_{f,Y}(x_0)$.*

Proof Since $f[(g_0 - g) + t(x_0 - g_0)]$ is a convex fonction, so $\tau_f(g_0 - g, x_0 - g_0) \ge \rho f(x_0 - g_0)$ implies that, for each $t > 0$,

$$\rho f(x_0 - g_0) \le f[(g_0 - g) + t(x_0 - g_0)] - f(g_0 - g).$$

This is the definition of $g_0 \in R^\rho_{f,Y}(x_0)$. □

Remark The inverse of Lemma 3.1 is not true in general, except when Y is a subspace of X.

Theorem 3.2. *Let Y be a subspace of X and f satisfy the condition (F2). Then $g_0 \in R^\rho_{f,Y}(x_0)$ if and only if, for every $g \in Y$, $\tau_f(g, x_0 - g_0) \ge \rho f(x_0 - g_0)$.*

Proof we need only to show the necessarify. Since $g_0 \in R^\rho_{f,Y}(x_0)$, for every $g \in Y$, we have

$$f(g - g_0) + \rho f(x - g_0) \le f(x - g).$$

Hence we have

$$f(g) + \rho f(x - g_0) \le f[(x - g_0) - g],$$

that is, $0 \in R^{\rho}_{f,Y}(x - g_0)$. Since f is a sublinear function, we have $0 \in R^{\rho}_{f,Y}[t(x_0 - g_0)]$ for each $t \ge 0$. This implies that, for every $t > 0$,

$$f(-g) + t\rho f(x - g_0) \le f[t(x - g_0) + g]. \tag{3.1}$$

Since f is symmetric, (3.1) implies that

$$\rho f(x - g_0) \le [f[t(x - g_0) + g] - f(g)/t.$$

Hence $\tau_f(g, x_0 - g_0) \ge \rho f(x_0 - g_0)$. □

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摘 要

研究了在局部凸空间中的 f- 余逼近和强 f- 余逼近的一些性质.