else (i.e., $v^{(i)} \in V_S(\triangle)$)

$$w_i = 0, \quad z_i = 0, \quad w_i' = \sigma_1^{(i)},$$

and in the later case (i.e., $v^{(i)} \in V_S(\triangle)$), all the elements in the $2i$-th row of matrix $A$ equal null since $2i$-th equation in (4.1) is in fact an identity.

Hence, we can see easily that the coefficient matrix $A$ in (4.2) can be transformed into an echelon matrix by exchanging its rows, and the rank of $A$ is $2\beta - \gamma$. Therefore we have

$$\dim \hat{S}_2^0(\triangle) = \dim S_2^0(\triangle) - \operatorname{rank}(A) = \alpha + \rho - (2\beta - \gamma).$$

Which together with theorem 2 gives the following main result.

**Theorem 3** *Let $\Omega$ be a simply connected polygonal region in $R^2$ and $\triangle$ be a triangulation of $\Omega$. If $(\Omega, \triangle)$ is type-$X$ and $\triangle$ is a stratified triangulation, then*

$$\dim S_3^1(\triangle) = \alpha + \rho - 2\beta + \gamma + 4.$$

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# 一类分层三角剖分下三次样条空间的维数

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## 摘 要

本文定义了平面单连通多边形域的一类较任意的三角剖分 —- 分层三角剖分，并通过分析二元样条的积分协调条件，确定了分层三角剖分下三次 $C^1$ 样条函数空间的维数.

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# The Dimension of Cubic Spline Space over Stratified Triangulation *

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**Abstract** In this paper, a kind of so-called stratified triangulation is defined. By analysing the integral conformality condition of bivariate spline, we determine the dimension of space of cubic splines over stratified triangulation.

**Keywords** integral conformality condition, cubic spline, stratified triangulation, dimension.

**Classification** AMS(1991) 65D07,41A63/CCL O241.5

## 1. Introduction

Let $\Omega \subset R^2$ be a connected polygonal domain, which does not contain any hole. Let $\triangle$ be a triangulation[1] of $\Omega$ and denote

$$\begin{aligned}
&V_I(\triangle) := \text{all the interior vertices of } \triangle\\
&V_B(\triangle) := \text{all the boundary vertices of } \triangle\\
&E_I(\triangle) := \text{all the interior edges of } \triangle\\
&E(\triangle) := \text{all the edges of } \triangle\\
&V(\triangle) := V_I(\triangle) \cup V_B(\triangle)
\end{aligned}$$

We call $(\Omega, \triangle)$ to be type-$X$ provided that there exists a rectangular coordinates system $XOY$ such that the slope of any edge of $\triangle$ in $XOY$ equals neither 0 not $\infty$ and the number of the intersection points of $\partial\Omega$ and any straight line which parallels $X$-axis is no more than two. For example, $(\Omega, \triangle)$ is always type-$X$ for any triangulation $\triangle$ when $\Omega$ is a connected convex polygonal region. We will merely consider the case that $(\Omega, \triangle)$ is type-$X$ throughout this paper.

Set up a rectangular coordinate system $XOY$ as required above. Clearly, the two boundary vertices of $\triangle$ with the maximum ordinate and the minimum ordinate divide $\partial\Omega$ into two parts. The left one is denoted by $\partial^-\Omega$ and the set of the vertices on $\partial^-\Omega$ is denoted by $V_B^-(\triangle)$ except for the end points.

Suppose the vertices in $V_I(\triangle)$ are labeled $v^{(i)}, i = 1, 2, \cdots, |V_I(\triangle)|$, here $|V_I(\triangle)|$ is the cardinality of $V_I(\triangle)$. For each $v^{(i)} \in V_I(\triangle)$, the union of all the triangles with

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the common vertex $v^{(i)}$ is called a vertex star and denoted by $U(v^{(i)})$. We denote the boundary vertices of $U(v^{(i)})$, in the counterclockweise direction, by $v_j^{(i)}, j = 1, 2, \cdots, d_i$, where $d_i$ is the number of the edges emanating from $v(i)$, and set $v^{(i)} = v_0^{(i)}, e_j^{(i)} = v_0^{(i)} v_j^{(i)}$ and $k_j^{(i)} = \text{slope}(e_j^{(i)})$. Clearly, $e_j^{(i)}$ can be described by the equation $x = l_j^{(i)}(y)$, where $l_j^{(i)}(y) := x_0^{(i)} + (y - y_0^{(i)})/k_j^{(i)}$ and $(x_0^{(i)}, y_0^{(i)})$ is the coordinate of $v_0^{(i)}$. In addition, the triangle between $e_j^{(i)}$ and $e_{j+1}^{(i)}$ is denoted by $T_j^{(i)}, j = 1, \cdots, d_i$.

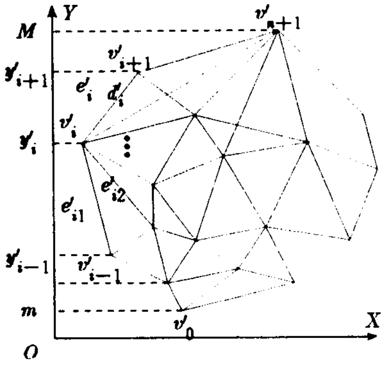


Figure 1

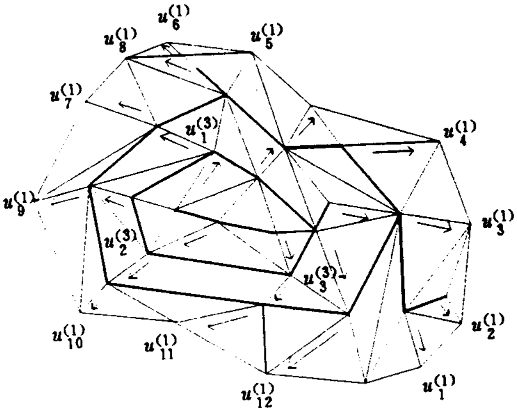

Figure 2

As shown in Fig.1, from bottom to top, we label consecutively all the vertices on $\partial^- \Omega$ $v_i', i = 0, 1, \cdots, n, n+1$. And for $i = 1, \cdots, n$, we number all the edges attached to $v_i'$ counterclockwise and label them $e_{ij}', j = 1, \cdots, d_i'$, here we have put $e_{i1}' = v_{i-1}' v_i'$. Note that the above-mentioned assumption of slope$(e) \neq 0, \infty$ for any $e \in E(\triangle)$, the equation of $e_{ij}'$ can be denoted by $x = \psi_{ij}(y), i = 1, \cdots, n, j = 1, \cdots, d_i'$. In addition, we let $T_{ij}'$ stand for the triangle between $e_{ij}'$ and $e_{ij+1}'$.

From the assumption that $(\Omega, \triangle)$ is type-$X$, we can see easily that the projections of vertices $v_i' (i = 0, 1, \cdots, n, n+1)$ in $Y$-axis form a univariate partition $\triangle_y$ of interval $[m, M]$ as follows

$$\triangle_y : m = y_0' < y_1' < \cdots < y_n' < y_{n+1}' = M,$$

where $y_i'$ is the ordinate of $v_i'$. We denote the equation of the broken line on which $\partial^- \Omega$ lies by $x = \Phi(y)$:

$$\Phi(y) := \begin{cases} \psi_{i1}(y), & y \in [y_{i-1}', y_i'), i = 1, \cdots, n, \\ \psi_{nd_n'}(y), & y \in [y_n', y_{n+1}']. \end{cases}$$

For $0 \leq r \leq k-1$, where $r$ and $k$ are integers, we define $S_k^r(\triangle)$ to be vector space of $C^r$ functions which are piecewise polynomials with total degree at most $k$ over each triangle of $\triangle$. The space $S_k^r(\triangle)$ is called a bivariate spline space.

Let $\prod_k$ denote the space of bivariate polynomials with total degree $k$. As is well known, a necessary and sufficient condition for $s \in S_k^r(\triangle)$ is that the following conformality condition[2]

$$\sum_{j=1}^{d_i} q_j^{(i)}(x,y)[x-l_j^{(i)}(y)]^{r+1} \equiv 0 \tag{1.1}$$

holds for any $v_0^{(i)} \in V_I(\triangle)$, where $q_j^{(i)} \in \prod_{k-r-1}$ is the smoothing cofactor of $s$ from $T_{j-1}^{(i)}$ to $T_j^{(i)}$ across $e_j^{(i)}$ and satisfying

$$s(x,y)|T_j^{(i)} - s(x,y)|T_j^{(i)} = q_j^{(i)}[x-l_j^{(i)}(y)]^{r+1}. \tag{1.2}$$

Recently, in [3], another new identity, namely:

$$\sum_{j=2}^{d_i} \int_{l_1^{(i)}(y)}^{l_j^{(i)}(y)} q_j^{(i)}(x,y)[x-l_j^{(i)}(y)]^{r+1}dx \equiv 0, \tag{1.3}$$

called the integral conformality condition of spline function $s$ at the interior vertex $v_0^{(i)}$, was introduced. And a kind of related spline space $\hat{S}_k^r(\triangle)$ was defined to be composed of $s$ such that

1) $s \in S_k^r(\triangle)$;

2) the integral conformality condition (1.3) holds at each $v_0^{(i)} \in V_I(\triangle)$.

On the basis of these, [3] obtained the following

**Theorem 1** *Let $1 \le r \le k-1$ and $(\Omega, \triangle)$ by type-$X$, then a necessary and sufficient condition for $s \in S_k^r(\triangle)$ is that $s$ can be expressed as*

$$s(x,y) = \xi(y) + \int_{\Phi(y)}^{x} \eta(x,y)dx, \tag{1.4}$$

*where $\eta \in \hat{S}_{k-1}^{r-1}(\triangle)$, and $\xi(y)$ is a univariate piecewise polynomial with degree $k$ on $\triangle_y$ such that*

$$\xi_i(y) = \xi_0(y) + \sum_{t=1}^{i}\sum_{j=1}^{d_i'-1} \int_{\psi_{ij}(y)}^{\psi_{ij+1}(y)} \eta_j^{(t)}(x,y)dx, \quad i = 1,\cdots,n,$$

*with $\xi_i(y) := \xi(y)|_{[y_i', y_{i+1}']}, i = 0,1,\cdots,n$, and $\eta_j^{(i)}(x,y) := \eta(x,y)|_{T_{ij}'}, i = 1,\cdots,n, j = 1,\cdots,d_i'-1$.*

**Theorem 2** *Let $\triangle$ be a triangulation of $\Omega$ such that $(\Omega,\triangle)$ is type-$X$, then*

$$\dim S_k^r(\triangle) = (k+1) + \dim \hat{S}_{k-1}^{r-1}(\triangle).$$

## 2. On the integral conformality condition for $S_2^0(\triangle)$

Let $v_0^{(i)} \in V_I(\triangle)$ and $s \in S_2^0(\triangle)$, according to above section, the integral conformality condition of $s$ at $v_0^{(i)}$ is that

$$\sum_{j=2}^{d_i} \int_{l_1^{(i)}(y)}^{l_j^{(i)}(y)} q_j^{(i)}(x,y)[x-l_j^{(i)}(y)]dx \equiv 0$$

holds, where $q_j^{(i)} \in \prod_{2-0-1}$ is the smoothing cofactor of $s$ from $T_{j-1}^{(i)}$ to $T_j^{(i)}$ across $e_j^{(i)}$. Setting $q_j^{(i)}(x,y) = \alpha_j^{(i)}x + \beta_j^{(i)}y + \gamma_j^{(i)}, j = 1,\cdots,d_i$, and substituting them into the above identity, we have

$$\sum_{j=1}^{d_i} \alpha_j^{(i)}(l_j^{(i)}(y))^3 + 3\sum_{j=1}^{d_i}(\beta_j^{(i)}y + \gamma_j^{(i)})(l_j^{(i)}(y))^2 \equiv 0. \tag{2.1}$$

Note that $l_j^{(i)}(y) = x_0^{(i)} + (y - y_0^{(i)})/k_j^{(i)}$, the equation (2.1) can be rewritten as a homogeneous system of linear equations. Indeed, equating the coefficients of the various powers of $y$ to zero, we have

$$\begin{cases} \sum_{j=1}^{d_i}[\alpha_j^{(i)}(k_j^{(i)})^{-3} + 3\beta_j^{(i)}(k_j^{(i)})^{-2}] = 0, \\ \sum_{j=1}^{d_i}[x_0^{(i)}\alpha_j^{(i)}(k_j^{(i)})^{-2} + (y_0^{(i)}\beta_j^{(i)} + \gamma_j^{(i)})(k_j^{(i)})^{-2} + 2x_0^{(i)}\beta_j^{(i)}(k_j^{(i)})^{-1}] = 0, \\ \sum_{j=1}^{d_i}[x_0^{(i)}\alpha_j^{(i)}(k_j^{(i)})^{-1} + x_0^{(i)}\beta_j^{(i)} + 2(y_0^{(i)}\beta_j^{(i)} + \gamma_j^{(i)}(k_j^{(i)})^{-1}] = 0, \\ \sum_{j=1}^{d_i}[(y_0^{(i)}\beta_j^{(i)} + \gamma_j^{(i)}] = 0. \end{cases} \tag{2.2}$$

Since $s \in S_2^0(\triangle), s$ must satisfy the conformality condition:

$$\sum_{j=1}^{d_i} q_j^{(i)}(x,y)[x - l_j^{(i)}(y)] \equiv 0,$$

that is,

$$\begin{cases} \sum_{j=1}^{d_i} \alpha_j^{(i)} = 0, \\ \sum_{j=1}^{d_i} \beta_j^{(i)}(k_j^{(i)})^{-1} = 0, \\ \sum_{j=1}^{d_i}[y_0^{(i)}\alpha_j^{(i)}(k_j^{(i)})^{-1} + \gamma_j^{(i)}] = 0, \\ \sum_{j=1}^{d_i}[x_0^{(i)}\beta_j^{(i)} + \gamma_j^{(i)}(k_j^{(i)})^{-1}] = 0, \\ \sum_{j=1}^{d_i}[x_0^{(i)}\gamma_j^{(i)} - y_0^{(i)}\gamma_j^{(i)}(k_j^{(i)})^{-1}] = 0. \end{cases}$$

By putting these relations into (2.2), we see that the last two equations in (2.2) are actually identities, so they are redundant. While the remainder two equations can be rewritten as

$$\begin{cases} \sum_{j=1}^{d_i}[\alpha_j^{(i)}(k_j^{(i)})^{-3} + 3\beta_j^{(i)}(k_j^{(i)})^{-2}] = 0, \\ \sum_{j=1}^{d_i}[x_0^{(i)}\alpha_j^{(i)}(k_j^{(i)})^{-2} + (y_0^{(i)}\beta_j^{(i)} + \gamma_j^{(i)})(k_j^{(i)})^{-2}] = 0. \end{cases} \tag{2.3}$$

Let

$$s|_{T_j^{(i)}} = a_j^{(i)}x^2 + b_j^{(i)}xy + c_j^{(i)}y^2 + d_j^{(i)}x + e_j^{(i)}y + f_j^{(i)}. \tag{2.4}$$

Then it follows from (1.2) that

$$\begin{cases} \alpha_j^{(i)} = \nabla a_j^{(i)}, \\ \beta_j^{(i)} = \nabla b_j^{(i)} + \nabla a_j^{(i)} \cdot (k_j^{(i)})^{-1}, \\ \gamma_j^{(i)} = \nabla a_j^{(i)} \cdot x_0^{(i)} - \nabla a_j^{(i)} \cdot (k_j^{(i)})^{-1} y_0^{(i)} + \nabla d_j^{(i)}. \end{cases} \tag{2.5}$$

Combining (2.3) and (2.5) yields

$$\begin{cases} \sum_{j=1}^{d_i}[4a_j^{(i)} \cdot \triangle(k_j^{(i)})^{-3} + 3b_j^{(i)} \cdot \triangle(k_j^{(i)})^{-2}] = 0, \\ \sum_{j=1}^{d_i}[2a_j^{(i)}x_0^{(i)} + b_j^{(i)}y_0^{(i)} + d_j^{(i)}] \cdot \triangle(k_j^{(i)})^{-2} = 0, \end{cases} \tag{2.6}$$

where $\triangle$ ($\nabla$) are the forward (backward) differencing operators with respect to $j$, respectively.

To simplify (2.6) further, we need the well-known $B$-net technique[4]. For $s \in S_2^0(\triangle)$ and $T_j^{(i)} \in \cup(v_0^{(i)})$, denote

$$\begin{cases} s_j^{(i)} := s(v_j^{(i)}), & j = 0, 1, \cdots, d_i, \\ p_j^{(i)} := s((v_0^{(i)} + v_j^{(i)})/2) - s_0^{(i)}/2 - s_j^{(i)}/2, & j = 1, \cdots, d_i, \\ g_j^{(i)} := s((v_j^{(i)} + v_{j+1}^{(i)})/2) - s_j^{(i)}/2 - s_{j+1}^{(i)}/2, & j = 1, \cdots, d_i. \end{cases}$$

$s_0^{(i)}, s_j^{(i)}, s_{j+1}^{(i)}, p_j^{(i)}, p_{j+1}^{(i)}$ and $g_j^{(i)}$ are called the $B$-net ordinates of $s$ with respect to $T_j^{(i)}$, where $j + 1 \mathrm{mod}(d_i)$. By means of these $B$-net ordinates, the restriction of $s$ on $T_j^{(i)}$ can be expressed as

$$\begin{aligned} s|_{T_j^{(i)}} = {} & s_0^{(i)}\frac{2!}{2!0!0!}[\lambda_0^{(i)}]^2 + s_j^{(i)}\frac{2!}{0!2!0!}[\lambda_j^{(i)}]^2 + s_{j+1}^{(i)}\frac{2!}{0!0!2!}[\lambda_{j+1}^{(i)}]^2 \\ & + p_j^{(i)}\frac{2!}{1!1!0!}\lambda_0^{(i)}\lambda_j^{(i)} + p_{j+1}^{(i)}\frac{2!}{1!0!1!}\lambda_0^{(i)}\lambda_{j+1}^{(i)} + g_j^{(i)}\frac{2!}{0!1!1!}\lambda_j^{(i)}\lambda_{j+1}^{(i)}, \end{aligned} \tag{2.7}$$

where $\lambda_0^{(i)}, \lambda_j^{(i)}, \lambda_{j+1}^{(i)}$ are the barycentric coordinates of $v = (x, y)$ and are defined by

$$\lambda_t^{(i)} := D_t^{[i,j]}/D^{[i,j]}, \quad t = 0, j, j+1,$$

where

$$D^{[i,j]} := \text{area}[v_0^{(i)}, v_j^{(i)}, v_{j+1}^{(i)}] := \frac{1}{2!}\begin{vmatrix} 1 & x_0^{(i)} & y_0^{(i)} \\ 1 & x_j^{(i)} & y_j^{(i)} \\ 1 & x_{j+1}^{(i)} & y_{j+1}^{(i)} \end{vmatrix}$$

while $D_t^{[i,j]}$ is the resulting determination by substituing the point $v_t^{(i)}$ with $v = (x, y)$ in $D^{[i,j]}$.

From (2.4) and (2.7), we have

$$\begin{aligned} 2a_j^{(i)}x_0^{(i)} + b_j^{(i)}y_0^{(i)} + d_j^{(i)} &= \frac{\partial s}{\partial x}\Big|_{(x_0^{(i)}, y_0^{(i)})} \\ &= 2[s_0^{(i)}(y_j^{(i)} - y_{j+1}^{(i)}) + p_j^{(i)}(y_{j+1}^{(i)} - y_0^{(i)}) + p_{j+1}^{(i)}(y_0^{(i)} - y_j^{(i)})]/D^{[i,j]}. \end{aligned}$$

From which and $D^{[i,j]} = -\frac{1}{2}(y_j^{(i)} - y_0^{(i)})(y_{j+1}^{(i)} - y_0^{(i)})\Delta(k_j^{(i)})^{-1}$, the second equation in (2.6) can be simplified as

$$\sum_{j=1}^{d_i} \omega_j^{(i)}[p_j^{(i)} - s_0^{(i)}] = 0, \tag{2.8}$$

where $\omega_j^{(i)} := [(k_{j+1}^{(i)})^{-1} - (k_{j-1}^{(i)})^{-1}]/(y_j^{(i)} - y_0^{(i)})$.

Similary, it is not difficult to simplify the first equation in (2.6) as

$$\sum_{j=1}^{d_i}[\mu_j^{(i)}(s_j^{(i)} + s_0^{(i)} - 2p_j^{(i)}) + \sigma_j^{(i)}(g_j^{(i)} - p_j^{(i)} - p_{j+1}^{(i)} + s_0^{(i)})] = 0, \tag{2.9}$$

where $\mu_j^{(i)} := [(k_{j+1}^{(i)})^{-1} - (k_{j-1}^{(i)})^{-1}]/(y_j^{(i)} - y_0^{(i)})^2$ and $\sigma_j^{(i)} := [(k_{j+1}^{(i)})^{-1} - (k_j^{(i)})^{-1}]/[(y_j^{(i)} - y_0^{(i)})(y_{j+1}^{(i)} - y_0^{(i)})]$.

Therefore, we have infered that, for $s \in S_2^0(\Delta)$, $s$ satisfies the integral conformality condition at the vertex $v_0^{(i)}$ if and only if (2.8) and (2.9) hold.

## 3. The stratified triangulation

For $v_0^{(i)} \in V_I(\Delta)$, if $d_i = 4$ and $k_j^{(i)} = k_{j+2}^{(i)}, j = 1, 2$, then $v_0^{(i)}$ is called a singular vertex of $\Delta$, and the set of all the singular vertices of $\Delta$ is denoted by $V_S(\Delta)$.

Let $v \in V(\Delta)$ and $V^* \subset V(\Delta)$, $v$ and $V^*$ are said to be adjacent if there exists some $u \in V^*$ such that $u$ and $v$ are adjacent. The set of the vertices in $\Delta$ which are adjacent to $V^*$ is denoted by $B(V^*)$. If $V^* = \{v\}, B(V^*)$ will be written in simplified form $B(v)$. It is clear that $B(V^*) = \cup_{v \in V^*} V_B(\cup(v))$ and $B(v) = V_B(\cup(v))$.

In view of the convenience of description, we need a few concepts and symbols in the theory of graphs. It is clear that the vertices and edges of the triangulation $\Delta$ form a connected planar graph, which will be denoted by $G$.

**Definition 1** *$G'$ is called a qusi-circuit in $G$ if $G'$ is a connected subgraph of graph $G$ such that*

*1) $E(G')$ contains a unique circuit, denoted by $C$;*

2) $|V_I(G')| = 0$ and $|E_I(G')| = 0$;

3) if $v \in V(G')\backslash V(C)$, then $|B(v) \cap V(C)| = 0$ or 1.

Clearly a circuit in $G$ must be a quasi-circuit in $G$. In what follows, we will denote a quasi-circuit by $C_q$, and call the unique circuit $C$ contained in $C_q$ the base circuit of $C_q$.

**Definition 2** *For $v_0^{(i)} \in V_I(\triangle)\backslash V_S(\triangle)$, if $v_j^{(i)} \in V_B(\cup(v_0^{(i)}))$ such that $k_{j-1}^{(i)} \neq k_{j+1}^{(i)}$, then $v_j^{(i)}$ is called a companion vertex of $v_0^{(i)}$. For $v_0^{(i)} \in V_S(\triangle)$, any $v_j^{(i)} \in V_B(\cup(v_0^{(i)}))$ is also called a companion vertex of $v_0^{(i)}$.*

**Definition 3** *Let $V_1$ and $V_2$ be two subsets of $V(\triangle)$ with the same cardinality $n$. $V_2$ is called a companion set of $V_1$ provided that all the vertices of $V_1$ and $V_2$ can be numbered as $u_1,\cdots,u_n$ and $v_1,\cdots,v_n$, respectively, such that*

*i) for $t = 1,\cdots,n, v_t$ is a companion vertex of $u_t$;*

*ii) if $n > 1$ then, for $t = 2,\cdots,n, v_{t-1} \notin \cup_{j=t}^{n} V_B(\cup(u_j))$.*

In addition, the collection of all the companion sets of $V_1$ is denoted by $\Theta(V_1)$.

To characterize the companion set more conveniently, here we introduce the notion of "companion matrix". Let $V_1 = \{u_1,\cdots,u_m\}$ and $V_2 = \{v_1,\cdots,v_n\} (n \geq m)$ be two subsets of $V(\triangle)$. The companion matrix $K(V_1;V_2) = (k_{ij})$ is an $m \times n$ matrix defined as follows:

$$k_{ij} = \begin{cases} 0, & \text{if } v_j \notin V_B(\cup(u_i)), \\ 1, & \text{if } v_j \text{ is a companion vertex of } u_i, \\ -1, & \text{otherwise.} \end{cases}$$

**Proposition 1** *Let $V_1, V_2 \subseteq V(\triangle)$ with $|V_1| = n$ and $|V_2| = m, m \geq n$. A necessary and sufficient condition for $\Xi(V_2) \cap \Theta(V_1) \neq \emptyset$ is that, there exists at least one $n \times n$ submatrix of $K(V_1;V_2)$ which, by exchanging the rows and columns, respectively, can be transformed into a upper triangular matrix with all the entries on main diagonal being 1, where $\Xi(V_2)$ is the collection of all the subset of $V_2$.*

We now give the definition of stratified triangulation.

**Definition 4** *Let $G$ be the graph consitsing of the vertices and edges of a triangulation $\triangle$ of $\Omega$. Set*

$$\begin{aligned} &V_B^0(\triangle) := V_B(\triangle), \\ &V_B^i(\triangle) := (V_I(\triangle) \cap B(V_B^{i-1}(\triangle)))\backslash \cup_{j=0}^{i-1} V_B^j(\triangle), \quad i = 1,2,\cdots. \end{aligned}$$

*$\triangle$ is called a stratified triangulation with $N$ layers if the following conditions are satisfied:*

*i) $V_I(\triangle) = \cup_{i=1}^{N} V_B^i(\triangle)$;*

*ii) for $i = 1,\cdots,N-1$, there exists a quasi-circuit $C_q^i \subseteq G$ with the base circuit $C_i$ such that $V_B^i(\triangle) = V(C_q^i)$, and for $i = 1,\cdots,N, V_B^i(\triangle) \subset \text{int}C_{i-1}$, where $C_0 = \partial\Omega$;*

*iii) for $i = 1,\cdots,N, \Xi(V(C_{i-1})) \cap \Theta(V_B^i(\triangle)) \neq \emptyset$.*

For example, the triangulation in Fig.2 is a stratified triangulation with three layers, where the two quasi-circuits $C_q^i, i = 1,2$, are drawn with thich lines, and the path on which the vertices of $V_B^3(\triangle)$ lie is also drawn with thich line. In addition, all the vertices of the companion sets of $V_B^1(\triangle)$ and $V_B^3(\triangle)$ are labeled as $u_t^{(i)}, i = 1,3, t = 1,\cdots,\tau_i$, respectively, with $\tau_1 = 12$ and $\tau_3 = 3$.

## 4. The dimension of $S_3^1(\triangle)$

Let $\triangle$ be a stratified triangulation with $N$ layers and $G$ be the graph consisting of the vertices and edges of $\triangle$. We denote the cardinalities of $V(\triangle), V_I(\triangle), V_S(\triangle)$ and $E(\triangle)$ by $\alpha, \beta, \gamma$ and $\rho$, respectively.

Since the case of $\beta = 1$ and the case that $\triangle$ contains flats as its triangles had been discussed in [1], we may assume that $\beta > 1$ and $\triangle$ contains no flat.

According to definition 4, $V_I(\triangle) = \cup_{i=1}^N V_B^i(\triangle)$ and for $i = 1,\cdots,N, \Xi(V(C_{i-1})) \cap \Theta(V_B^i(\triangle)) \neq \emptyset$, where $C_i$ is the base circuit of $C_q^i$. Hence, if we let $V_B^i(\triangle) = \{z_1^{(i)},\cdots,z_{\tau_i}^{(i)}\}$, then there exists at least one subset of $V(C_{i-1})$, denoted by $\{w_i^{(i)},\cdots,w_{\tau_i}^{(i)}\}$, such that $\{w_1^{(i)},\cdots,w_{\tau_i}^{(i)}\}$ is a companion set of $V_B^i(\triangle)$. Clearly, without loss of generality, we can assume that the companion matrix $K(\{z_1^{(i)},\cdots,z_{\tau_i}^{(i)}\} : (w_1^{(i)},\cdots,w_{\tau_i}^{(i)}\})$ is a upper triangular matrix with $k_{tt} = 1, t = 1,\cdots,\tau_i$.

We now renumber the vertices in $V_I(\triangle)$ and still denote them by $v^{(i)})$ or $v_0^{(i)}, i = 1,\cdots,\beta$, but provided that the following one-to-one correspondence holds

$$\begin{array}{cccccccccc} z_1^{(1)}, & \cdots, & z_{\tau_1}^{(1)}; & z_1^{(2)}, & \cdots, & z_{\tau_2}^{(2)}; & \cdots; & z_1^{(N)}, & \cdots, & z_{\tau_N}^{(N)} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ v^{(1)}, & \cdots, & v^{(\theta_1)}; & v^{(\theta_1+1)}, & \cdots, & v^{(\theta_2)}; & \cdots; & z^{(\theta_{N-1}+1)}, & \cdots, & v^{(\theta_N)} \end{array}$$

where $\theta_j = \sum_{k=1}^j \tau_k, j = 1,\cdots,N$, and $\theta_N = \beta$. Hence $V_I(\triangle) = (v^{(1)},\cdots,v^{(\beta)}\} = \{v_0^{(1)},\cdots,v_0^{(\beta)}\}$. Similarly, we renumber $\{w_1^{(1)},\cdots,w_{\tau_1}^{(1)};\cdots;w_1^{(N)},\cdots,w_{\tau_N}^{(N)}\}$ as $\{w_1,\cdots,w_\beta\}$. Then the companion matrix $K(\{v^{(1)},\cdots,v^{(\beta)}\};\{w_1,\cdots,w_\beta\})$ is a upper triangular matrix with $k_{ii} = 1, i = 1,\cdots,\beta$.

From (2.8) and (2.9), we have that a necessary and sufficient condition for $s \in \hat{S}_2^0(\triangle)$ is that $s \in S_2^0(\triangle)$ and all the free parameters $s_j^{(i)}, i = 1,\cdots,\beta, j = 0,1,\cdots,d_i, p_j^{(i)}$ and $g_j^{(i)}, i = 1,\cdots,\beta, j = 1,\cdots,d_i$ (for the reason of overlapping, the total number of them is actually $\alpha + \rho$) of $S_2^0(\triangle)$ must satisfy the linear system:

$$\begin{cases} \sum_{j=1}^{d_i}[\mu_j^{(i)}(s_j^{(i)} + s_0^{(i)} - 2p_j^{(i)}) + \sigma_j^{(i)}(g_j^{(i)} - p_j^{(i)} - p_{j+1}^{(i)} + s_0^{(i)})] = 0, & i = 1,\cdots,\beta, \\ \sum_{j=1}^{d_i}\omega_j^{(i)}[p_j^{(i)} - s_0^{(i)}] = 0, & i = 1,\cdots,\beta. \end{cases} \tag{4.1}$$

Note that $v_1^{(i)} = w_i$ is a companion vertex of $v_0^{(i)}$, hence, when $v_0^{(i)} \in V_I(\triangle)\backslash V_S(\triangle)$, the coefficient $\mu_1^{(i)}$ of term $s_1^{(i)}$ in the $(2i-1)$-th equation and the coefficient $\omega_1^{(i)}$ of term $p_1^{(i)}$ in the $2i$-th equation of (4.1) are nonzero; when $v_0^{(i)} \in V_S(\triangle)$, the $2i$-th equation of (4.1) is actually an identity and the $(2i-1)$-th equation can be rewritten as

$$\sum_{j=1}^{4}\sigma_j^{(i)}[g_j^{(i)} - p_j^{(i)} - p_{j+1}^{(i)} + s_0^{(i)}] = 0,$$

in which, the coefficient $\sigma_1^{(i)}$ of term $g_1^{(i)}$ is also nonzero.

Since $K(\{v^{(1)},\cdots,v^{(\beta)}\};\{w_1,\cdots,w_\beta\})$ is a upper triangular matrix with $k_{tt}=1, t=1,\cdots,\beta$, and $v_1^{(t)}=w_t$ is a companion vertex of $v^{(t)}$, hence, it follows from definition 3 that

$$v_1^{(t)} \notin \cup_{j=t+1}^{\beta} V(\cup(v^{(j)})), \quad t=1,\cdots,\beta-1,$$
$$e_1^{(t)} \notin \cup_{j=t+1}^{\beta} E(\cup(v^{(j)})), \quad t=1,\cdots,\beta-1,$$

and

$$v_1^{(t)}v_2^{(t)} \notin \cup_{j=t+1}^{\beta} E(\cup(v^{(j)})), \quad t=1,\cdots,\beta-1.$$

That is to say, $s_1^{(t)}, p_1^{(t)}$ and $g_1^{(t)}$ will not apper in any equation in (4.1) for $i=t+1,\cdots,\beta$. Hence, (4.1) can be rewritten as

$$AX=0, \tag{4.2}$$

where $X=(x_1,x_2,\cdots,x_{2\beta-1},x_{2\beta};x_{2\beta+1},\cdots,x_{\alpha+\rho})^T$ provided that, for $i=1,\cdots,\beta$, if $v^{(i)}\in V_I(\triangle)\backslash V_S(\triangle)$ then $x_{2i-1}=s_1^{(i)}$ and $x_{2i}=p_1^{(i)}$ else (i.e., $v^{(i)}\in V_S(\triangle)$)$x_{2i-1}=s_1^{(i)}$ and $x_{2i}=g_1^{(i)}$. While $x_j, j=2\beta+1,\cdots,\alpha+\rho$ are the remainder $\alpha+\rho-2\beta$ $B$-net ordinates of $s(x,y)$ on $\triangle$, which can be arranged in free order. And

$$A=(H,Q),$$

where $Q$ is an $(2\beta)\times(\alpha+\rho-2\beta)$ matrix and $H$ is an $2\beta\times 2\beta$ matrix of the form

$$H=\begin{bmatrix} E_1 & F_1 & G_1 & & & & \\ & E_2 & F_2 & G_2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & E_{N-2} & F_{N-2} & G_{N-1} \\ & & & & & E_{N-1} & F_{N-1} \\ & & & & & & E_N \end{bmatrix}$$

with $F_j(j=1,\cdots,N-1)$ and $G_j(j=1,\cdots,N-2)$ being $2\tau_j\times 2\tau_{j+1}$ and $2\tau_j\times 2\tau_{j+2}$ blocks, respectively, and for $j=0,1,\cdots,N-1$,

$$E_{j+1}=\begin{bmatrix} w_{\theta_j+1} & w'_{\theta_j+1} & & & & & & L_{j+1} \\ & z_{\theta_j+1} & 0 & & & & & \\ & & w_{\theta_j+2} & w'_{\theta_j+2} & & & & \\ & & & z_{\theta_j+2} & 0 & & & \\ & & & & \ddots & \ddots & & \\ & & & & & & w_{\theta_{j+1}} & w'_{\theta_{j+1}} \\ & & & & & & & z_{\theta_{j+1}} \end{bmatrix}$$

with $L_{j+1}$ being an $(\tau_{j+1}-2)\times(\tau_{j+1}-2)$ upper triangular block, and for $i=1,\cdots,\beta$, if $v^{(i)}\in V_I(\triangle)\backslash V_S(\triangle)$ then

$$w_i=\mu_1^{(i)},\quad w'_i=-2\mu_1^{(i)}-\sigma_1^{(i)}-\sigma_{d_i}^{(i)},\quad z_i=\omega_1^{(i)},$$

else (i.e., $v^{(i)} \in V_S(\triangle)$)

$$w_i = 0, \quad z_i = 0, \quad w_i' = \sigma_1^{(i)},$$

and in the later case (i.e., $v^{(i)} \in V_S(\triangle)$), all the elements in the $2i$-th row of matrix $A$ equal null since $2i$-th equation in (4.1) is in fact an identity.

Hence, we can see easily that the coefficient matrix $A$ in (4.2) can be transformed into an echelon matrix by exchanging its rows, and the rank of $A$ is $2\beta - \gamma$. Therefore we have

$$\dim \hat{S}_2^0(\triangle) = \dim S_2^0(\triangle) - \mathrm{rank}(A) = \alpha + \rho - (2\beta - \gamma).$$

Which together with theorem 2 gives the following main result.

**Theorem 3** *Let $\Omega$ be a simply connected polygonal region in $R^2$ and $\triangle$ be a triangulation of $\Omega$. If $(\Omega, \triangle)$ is type-$X$ and $\triangle$ is a stratified triangulation, then*

$$\dim S_3^1(\triangle) = \alpha + \rho - 2\beta + \gamma + 4.$$

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# 一类分层三角剖分下三次样条空间的维数

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## 摘 要

本文定义了平面单连通多边形域的一类较任意的三角剖分 —- 分层三角剖分，并通过分析二元样条的积分协调条件，确定了分层三角剖分下三次 $C^1$ 样条函数空间的维数.