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Remark on Two Results of Nunokawa *

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Abstract The object of this paper is to give an answer to the question by M.Nunokawa[1].

Keywords starlike and convex function, p-valently functions, subordinate.

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1. Introduction

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad p \in N = \{1,2,3,\cdots\} \tag{1.1}$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$.

A function $f(z)$ belonging to the class $A(p)$ is said to be p-valently starlike in E if it satisfies

$$\mathrm{Re}\{\frac{zf'(z)}{f(z)}\} > 0 \quad p \in E. \tag{1.2}$$

Denote by $S^*(p)$ the class of p-valently starlike functions in E.

A function $f(z)$ in $A(p)$ is said to be p-valently convex if it satisfies

$$\mathrm{Re}\{1 + \frac{zf''(z)}{f'(z)}\} > 0 \quad z \in E. \tag{1.3}$$

Denote by $C(p)$ the class of p-valently convex functions in E.

Nunokawa[1] proved that

Theorem A *Let $f(z) \in A(p)$ and suppose that*

$$|1 + \frac{zf''(z)}{f'(z)}| < \frac{1}{p}\log(4e^{p-1})|\frac{zf'(z)}{f(z)}|, \quad z \in E. \tag{1.4}$$

Then $f(z) \in S^*(p)$.

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Theorem B *Let $g(z) \in A(p)$ and suppose that*

$$|1 + \frac{zg^{(p+1)}(z)}{g^{(p)}(z)}| < \log 4 |\frac{zg^{(p)}(z)}{g^{(p-1)}(z)}|, \quad z \in E. \tag{1.5}$$

Then $g(z) \in S^*(p)$.

In view of Theorem A and Theorem B, Nunokawa[1] asked the question whether Theorem A and Theorem B are sharp or not.

In this paper, we show that Theorem A and Theorem B are not sharp.

2. Preliminary

For our purpose, we require the following lemmas.

Lemma 1[2] *Let $q(z)$ be univalent in E and let $\theta(w)$ and $\Phi(w)$ be analytic in a domain D containing $q(E)$, with $\Phi(w) \neq 0$ when $w \in q(E)$. Set $Q(z) = zq'(z)\Phi(q(z)), h(z) = \theta(q(z)) + Q(z)$ and suppose that*

(1) $Q(z)$ is starlike(univalent) in E,

and

(2) $\mathrm{Re}\frac{zh'(z)}{Q(z)} = \mathrm{Re}\{\frac{\theta'(q(z))}{\Phi(q(z))} + \frac{zQ'(z)}{Q(z)}\} > 0, \quad z \in E.$

If $p(z)$ is analytic in E, with $p(0) = q(0), p(E) \subset D$ and

$$\theta(p(z)) + zp'(z)\Phi(p(z)) \prec \theta(q(z)) + zq'(z)\Phi(q(z)) = h(z), \tag{2.1}$$

then $p(z) \prec q(z)$, and $q(z)$ is the best dominant of (2.1).

Lemma 2 *Let $p(z)$ be analytic in E with $p(0) = 1$, and $p(z) \neq 0$ for $0 < |z| < 1$. If $p(z)$ satisfies*

$$1 + \frac{zp'(z)}{p \cdot p^2(z)} \prec 1 + \frac{2z}{p(1+z)^2} = h(z), \tag{2.2}$$

then $p(z) \prec \frac{1+z}{1-z}$.

Proof If we take $q(z) = \frac{1-z}{1+z}, \theta(w) = 1$ and $\Phi(w) = \frac{1}{pw^2}$ in lemma 1, then it is easy to show that $q(z), \theta(w)$ and $\Phi(w)$ satisfy the conditions of lemma 1. Since

$$Q(z) = zq'(z)\Phi(q(z)) = \frac{2z}{p(1+z)^2}$$

is starlike in E and

$$h(z) = \theta(q(z)) + Q(z) = 1 + \frac{2z}{p(1+z)^2},$$

it may be readily checked that the condition (1) and (2) of Lemma 1 are satisfied, so the result follows from (2.1). □

Lemma 3[3] *Let $f(z) \in A(p)$. Suppose that there exists a positive integer k for which*

$$k + \mathrm{Re}\frac{zf^{(k+1)}(z)}{f^{(k)}(z)} > 0, \quad z \in E,$$

where $1 \le k \le p$. Then we have

$$k - 1 + \mathrm{Re}\frac{zf^{(k)}(z)}{f^{(k-1)}(z)} > 0, \quad z \in E.$$

3. Main Results

Theorem 1 *Let $f(z) \in A(p)$ with $f(z) \cdot f'(z) \neq 0$ for $0 < |z| < 1$. If*

$$|1 + \frac{zf''(z)}{f'(z)}| < (1 + \frac{1}{2p})|\frac{zf'(z)}{f(z)}|, \quad z \in E, \tag{3.1}$$

then $f(z) \in S^*(p)$.

Proof Let $p(z) = \frac{zf'(z)}{pf(z)}$. Then $p(z)$ is analytic in E with $p(0) = 1$ and $p(z) \neq 0$ for $z \in E$. From (3.1), we get

$$\begin{aligned} &|\frac{f(z)}{zf'(z)} + \frac{f(z)f''(z)}{(f'(z))^2}| = |\frac{f(z)}{zf'(z)} + (z - \frac{f(z)}{f'(z)})'| = |\frac{1}{p \cdot p(z)} + (z - \frac{z}{p \cdot p(z)})'| \\ &= |1 + \frac{zp'(z)}{p \cdot p^2(z)}| < 1 + \frac{1}{2p}, \quad z \in E, \end{aligned}$$

which implies that

$$1 + \frac{zp'(z)}{p \cdot p^2(z)} \prec 1 + \frac{2z}{p(1+z)^2}. \tag{3.2}$$

Using lemma 2 and (3.2), we have

$$p(z) = \frac{zf'(z)}{pf(z)} \prec q(z) = \frac{1+z}{1-z}.$$

Then $f(z) \in S^*(p)$. □

From Theorem 1, we easily have the following corollary.

Corollary 1 *Let $f(z) \in A(1)$ and suppose that*

$$|1 + \frac{zf''(z)}{f'(z)}| < \frac{3}{2}|\frac{zf'(z)}{f(z)}|, \quad z \in E. \tag{3.3}$$

Then $f(z) \in S^*(1)$.

Theorem 2 *Let $g(z) \in A(p)$ with $g(z) \cdot g'(z) \neq 0$ for $0 < |z| < 1$. If*

$$|1 + \frac{zg^{(p+1)}(z)}{g^{(p)}(z)}| < \frac{3}{2}|\frac{zg^{(p)}(z)}{g^{(p-1)}(z)}|, \quad z \in E, \tag{3.4}$$

then $g(z) \in S^*(p)$. In particular, if $p \ge 2$, then $g(z) \in C(p)$.

Proof Let

$$f(z) = \frac{g^{(p-1)}(z)}{p!},$$

then $f(z) \in A(1)$. From the condition (3.4), we see that $f(z)$ satisfies (3.3). Using Corollary 1, we obtain

$$\mathrm{Re}\frac{zf'(z)}{f(z)} = \mathrm{Re}\frac{zg^{(p)}(z)}{g^{(p-1)}(z)} > 0, \quad z \in E.$$

From Lemma 3, we have

$$\mathrm{Re}\frac{zg'(z)}{g(z)} > 0, \quad z \in E.$$

In particular, if $p \geq 2$, we have by Lemma 3 that

$$\mathrm{Re}\{1 + \frac{zg''(z)}{g'(z)}\} > 0, \quad z \in E.$$

This completes the proof. □

References

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关于Nunokawa的两个结果

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摘 要

本文回答了 Nunokawa 在文 [1] 中提出的两个问题.