Journal of Mathematical Research & Exposition
Vol.16, No.4, 526–530, Nov. 1996

Spectral Mapping Theorems for C-Semigroups *

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Abstract By investigating the spectral characteristics for C-semigroups, we obtain a series of spectral mapping theorems and extend some of the theorems to the case of n-times integrated semigroups.

Keywords C-semigroup, integrated semigroup, spectral mapping theorem.

Classification AMS(1991) 47D05/CCL O177.1

1. Preliminaries

In recent years, the theoretical researches and application developments of operator semigroups have been made with remarkable speed. For example, motivated by the abstract Cauchy problem, two extensions of strongly continuous semigroups have appeared: C-semigroups and n-times integrated semigroups[1–8]. The purpose of this paper is to establish some spectral theorems for C-semigroups and integrated semigroups.

Let $(X, \|\cdot\|)$ be a complex Banach space, and $B(X)$ be a Banach algebra with the elements of bounded linear operators on X. For $T \in B(X)$, we use $\mathcal{D}(T), \mathcal{R}(T), \sigma(T), \sigma_p(T)$, $\sigma_{ap}(T), \rho(T), R(\lambda, T)$ for $\lambda \in \rho(T)$ as usual meaning. For $C \in B(X), \{T(t)\}_{t\geq 0} \subset B(X)$, if C is injective and $\mathcal{R}(C)$ is dense in X, and $\{T(t)\}$ satisfy: i) $T(t)T(s) = CT(t+s)$ for $t, s \geq 0$ and $T(0) = C$, ii) $T(\cdot)x : [0, +\infty) \to X$ is strongly continuous, iii) there exist $M > 0$ and $\eta \in R^1$ such that $\|T(t)\| \leq Me^{\eta t}$ for $t \geq 0$, then $\{T(t)\}$ is called C-semigroup. For C-semigroup $\{T(t)\}$, we define A as a generator of $\{T(t)\}$ as follows:

$$\begin{aligned} \mathcal{D}(A) &= \{x \in X : \lim_{t\to 0^+} \frac{1}{t}[T(t)x - Cx] \in \mathcal{R}(C)\}, \\ Ax &= C^{-1} \lim_{t\to 0^+} \frac{1}{t}[T(t)x - Cx], \quad \forall x \in \mathcal{D}(A). \end{aligned}$$

It is known that[3,4] when $\rho(A) \supset \{z \in \mathbf{C} : \mathrm{Re} z > \eta\}$ and A is densely-defined then

$$R(\lambda, A)x = C^{-1} \int_0^{+\infty} e^{-\lambda t} T(t)x dt, \quad x \in X, \tag{1}$$

*Received Mar.28, 1994. Project Supported by Coal Science Fundation of China.

$$\frac{dT(t)x}{dt} = T(t)Ax = AT(t)x, \quad x \in \mathcal{D}(A). \tag{2}$$

For a closed linear operator A on X, if there exists an $n \in \{0,1,2,\cdots\}$, constants M, η and a strongly continuous family $\{S(t)\}_{t\geq 0} \subset B(X)$ with $\|S(t)\| \leq Me^{\eta t}(t \geq 0)$ such that $\rho(A) \supset (\eta, +\infty)$ and

$$(\lambda - A)^{-1} = \lambda^n \int_0^{+\infty} e^{-\lambda t} S(t)dt, \quad \lambda > \eta, \tag{3}$$

then we say A generates the n-times integrated semigroup $\{S(t)\}$.

Lemma 1 *Suppose A generates the C-semigroup $\{T(t)\}$. Then*

$$Cx - e^{-\lambda t}T(t)x = (\lambda - A)\int_0^t e^{-\lambda s}T(s)xds, \quad x \in X. \tag{4}$$

Proof A simple integrating by parts and (2) yield

$$\begin{aligned}\int_0^t e^{-\lambda s}T(s)xds &= -\frac{1}{\lambda}e^{-\lambda s}T(s)x\Big|_0^t + \frac{1}{\lambda}A\int_0^t e^{-\lambda s}T(s)xds, \\ &= -\frac{1}{\lambda}e^{-\lambda t}T(t)x + \frac{1}{\lambda}Cx + \frac{1}{\lambda}A\int_0^t e^{-\lambda s}T(s)xds,\end{aligned}$$

which is the same as (4).

Lemma 2[5] *Suppose $\rho(A) \supset [0, +\infty), n \in \{0,1,2,\cdots\}$, then A generates the n-times integrated semigroup $\{S(t)\}$ if and only if A generates $C \equiv A^{-n}$-semigroup*

$$T(t) = S(t) + \sum_{j=0}^{n-1} \frac{t^j}{j!}A^{j-n}.$$

2. Main Theorems

Theorem 1 *Suppose $\{T(t)\}_{t\geq 0}$ is a C-semigroup with generator A densely-defined. If $e^{\lambda t} \in \rho(C^{-1}T(t)), t > 0$, then $\mu_k \in \rho(A)$ for $k \in N = \{0, \pm 1, \pm 2, \cdots\}$ and*

$$\sup_{k\in N} \|R(\mu_k, A)\| = M < +\infty, \tag{5}$$

where $\mu_k = \lambda + 2k\pi i/t$ and $R(\mu_k, A) = (\mu_k - A)^{-1}$.

Proof For $k \in N = \{0, \pm 1, \pm 2, \cdots\}$, obviously $e^{-\lambda t} = e^{-\mu_k t}$, if we substitute λ with μ_k in (4) and multiply $e^{\mu_k t}$ throughout (4), we have

$$(\mu_k - A)e^{\mu_k t}\int_0^t e^{-\mu_k t}T(s)xds = e^{\mu_k t}Cx - T(t)x = e^{\lambda t}Cx - T(t)x, \; x \in X. \tag{6}$$

Since $e^{\lambda t} \in \rho(C^{-1}T(t))$ for $t > 0$, therefore the inverse operator $(e^{\lambda t}C - T(t))^{-1}$ exists, left-multiplied by $(e^{\lambda t}C - T(t))^{-1}$ in (6) yields

$$(e^{\lambda t}C - T(t))^{-1}(\mu_k - A)e^{\mu_k t}\int_0^t e^{-\mu_k s}T(s)x ds = x, \quad x \in X. \tag{7}$$

Noticing that $T(t)Ax = AT(t)x, x \in \overline{\mathcal{D}(A)} = X$, we obtain

$$(\mu_k - A)e^{\mu_k t}(e^{\lambda t}C - T(t))^{-1}\int_0^t e^{-\mu_k s}T(s)x ds = x, \quad x \in X,$$

which shows that $e^{\mu_k t}(e^{\lambda t}C - T(t))^{-1}\int_0^t e^{-\mu_k s}T(s)ds$ is a right inverse operator of $(\mu_k - A)$. Similary, from (7) we can get

$$e^{\mu_k t}(e^{\lambda t}C - T(t))^{-1}\int_0^t e^{-\mu_k s}T(s)ds(\mu_k - A)x = x, \quad x \in X,$$

which shows that $e^{\mu_k t}(e^{\lambda t}C - T(t))^{-1}\int_0^t e^{-\mu_k s}T(s)ds$ is a left inverse of $(\mu_k - A)$. Therefore $\mu_k \in \rho(A)$ and

$$(\mu_k - A)^{-1} = R(\mu_k, A) = e^{\mu_k t}(e^{\lambda t}C - T(t))^{-1}\int_0^t e^{-\mu_k s}T(s)ds, \quad k \in N.$$

Now we come to estimate $\|R(\mu_k, A)\|$. Consider two cases:

i) $\mathrm{Re}\lambda = 0$.

In this case, $|\int_0^t e^{-\mu_k s}ds| \le \int_0^t |e^{-(\mathrm{Im}\lambda + \frac{2k\pi}{t})si}|ds \le \int_0^t 1ds = t$, so we have

$$\begin{aligned}
\|R(\mu_k, A)\| &= \|e^{\mu_k t}(e^{\lambda t}C - T(t))^{-1}\int_0^t e^{-\mu_k s}T(s)ds\| \\
&\le \|(e^{\lambda t}C - T(t))^{-1}\| \cdot |\int_0^t e^{-\mu_k s}ds| \cdot \sup_{0 \le s \le t}\|T(s)\| \\
&\le \|(e^{\lambda t}C - T(t))^{-1}\| \cdot t \cdot \sup_{0 \le s \le t}\|T(s)\| \triangleq M_1 < +\infty, \ \forall k \in N.
\end{aligned}$$

ii) $\mathrm{Re}\lambda \ne 0$,

In this case, $|\int_0^t e^{-\mu_k s}ds| \le \int_0^t e^{-\mathrm{Re}\lambda s}ds \le |\frac{e^{-\lambda t}+1}{\mathrm{Re}\lambda}|$, so we have

$$\begin{aligned}
\|R(\mu_k, A)\| &= \|e^{\mu_k t}(e^{\lambda t}C - T(t))^{-1}\int_0^t e^{-\mu_k s}T(s)ds\| \\
&\le e^{\mathrm{Re}\lambda t}\|(e^{\lambda t}C - T(t))^{-1}\| \cdot |\frac{1 + e^{-\lambda t}}{\mathrm{Re}\lambda}| \cdot \sup_{0 \le s \le t}\|T(s)\| \\
&\triangleq M_2 < +\infty, \quad \forall k \in N.
\end{aligned}$$

The above two steps show that (5) holds by taking $M = \max\{M_1, M_2\}$. □

Theorem 2 *Suppose $\{T(t)\}$ is a C-semigroup with generator A densely-defined, then*

$$\sigma(C^{-1}T(t)) \supset \{e^{\lambda t} : \lambda \in \sigma(A)\} \triangleq e^{t\sigma(A)}, \quad t > 0. \tag{8}$$

Proof By taking $k = 0$ in theorem 1, we know that $\lambda \in \rho(A)$ if $e^{\lambda t} \in \rho(C^{-1}T(t))$. This fact tells us $\rho(C^{-1}T(t)) \subset \{e^{\lambda t} : \lambda \in \rho(A)\}$. Since spectral set and resolvent set are not intersectable, so (8) holds. □

Theorem 3 *For C-semigroup $\{T(t)\}$ with generator A, we have*

(a) $e^{t\sigma_{ap}(A)} \subset \sigma_{ap}(C^{-1}T(t)), t \geq 0$;

(b) $e^{t\sigma_p(A)} = \sigma_p(C^{-1}T(t)) - \{0\}, t \geq 0$;

(c) $e^{t(\sigma_p(A)\cup\sigma_r(A))} = \sigma_p(C^{-1}T(t)) \cup \sigma_r(C^{-1}T(t)) - \{0\}, t \geq 0$.

Proof (a) Let $t \geq 0, \lambda \in \sigma_{ap}(A)$, then there exist $x_n \in X$ with $\|x_n\| = 1, n = 1, 2, \cdots$, such that $(\lambda - A)x_n \to 0$. From (4) we get

$$(C - e^{-\lambda t}T(t))x_n = \int_0^t e^{-\lambda s}T(s)(\lambda - A)x_n ds \to 0,$$

which shows that $e^{\lambda t} \in \sigma_{ap}(C^{-1}T(t))$ and (a) holds.

(b) From the relation[8] $\mathcal{N}(\lambda - A) = \cap_{t\geq 0}\mathcal{N}(Ce^{\lambda t} - T(t))$ we get

$$\begin{aligned} \forall\lambda \in \sigma_p(A) &\iff \exists x \neq 0 \text{ such that } (\lambda - A)x = 0 \Leftrightarrow x \in \mathcal{N}(\lambda - A) \\ &\iff \forall t \geq 0, x \in \mathcal{N}(e^{\lambda t} - C^{-1}T(t)) \Leftrightarrow (e^{\lambda t} - C^{-1}T(t))x = 0 \\ &\iff e^{\lambda t} \in \sigma_p(C^{-1}T(t)) - \{0\}. \end{aligned}$$

(c) Let $T^*(t)$ be the adjoint of $T(t)$ in the dual space $(X, \|\cdot\|)$, then $\{T^*(t)\}$ is also a semigroup of exponentially bounded linear operators and $T^*(0) = C$, but, as we have known in C_0 case, it is not strongly continuous generally. We define $X^{\odot}$ to be the subspace of X^* on which $T^*(t)$ is strongly continuous: $X^{\odot} = \{x^* \in X^* : \lim_{t\to 0^+} \|T^*(t)x^* - T^*(0)x^*\| = 0\}$. Let $T^{\odot}(t) = T^*(t)|_{X^{\odot}}$, then $\{T^{\odot}(t)\}_{t\geq 0}$ defines a C-semigroup on $X^{\odot}$ and its generator $A^{\odot}$ is the part of A^* in $X^{\odot}$. Since[4]

$$\sigma_r(A) \subset \sigma_p(A^{\odot}) \subset \sigma_r(A) \cup \sigma_p(A), \tag{9}$$

therefore

$$\sigma_r(C^{-1}T(t)) \subset \sigma_p(C^{-1}T^{\odot}(t)) \subset \sigma_r(C^{-1}T(t)) \cup \sigma_p(C^{-1}T(t)), \quad t \geq 0. \tag{10}$$

From (b) we have $\sigma_p(C^{-1}T^{\odot}(t) - \{0\} = e^{t\sigma_p(A^{\odot})}$, by which and (9) and (10) we conclude the results immediately.

Theorem 4 *Suppose $\{S(t)\}$ is an n-times integrated semigroup with generator A densely-defined and $\rho(A) \supset [0, +\infty), n \in \{0, 1, 2, \cdots\}$, then*

$$\sigma(S(t)) \supset \{e^{\lambda t}\lambda^{-n} - \sum_{j=0}^{n-1} \frac{t^j}{j!}\lambda^{j-n} : \lambda \in \sigma(A)\}, t \geq 0.$$

Proof Since A generates n-times integrated semigroup $\{S(t)\}$, A also generates A^{-n}-semigroup $\{T(t)\}$ by lemma 2 and

$$T(t) = S(t) + \sum_{j=0}^{n-1} \frac{t^j}{j!} A^{j-n},$$

by taking $C = A^{-n}$ in the theorem 2, and using some basic results on operator spectral theory, we obtain the result at once.

Theorem 2 and Theorem 3 can be called spectral mapping theorems for C-semigroups and Theorem 4 can be called spectral mapping theorem for n-times integrated semigroups.

Acknowledgement The author wants to thank Professor Yan Shaozong of Fudan University for his help and guidence.

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C 半群的谱映射定理

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摘要

本文从考察 C 半群的谱特征入手，得到了 C 半群的谱映射定理，并借助于 C 半群与积分半群的关系得到 n 次积分半群的一个谱映射定理.