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关于 Hurwitz Zeta-函数导数的一类新型均值公式*

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摘　要　本文利用解析方法给出 $\zeta'(s,\alpha)$ 对参数 α 的二次积分均值的一个很强的渐近公式

关键词　Hurwitz Zeta-函数, 均值, 渐近公式

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§1　引言及引理

对于实数 $0 < \alpha \leq 1$, 设 $\zeta(s,\alpha)$ 表示 Hurwitz Zeta-函数, 当 $\mathrm{Re}\, s > 1$ 时定义 $\zeta(s,\alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}$, $\zeta_1'(s,\alpha)$ 表示 $\zeta(s,\alpha)$ 关于复变量 s 的一阶导数, $\zeta'(s,\alpha) = \zeta_1'(s,\alpha) + \alpha^{-s}\ln\alpha$. 本文的主要目的是讨论均值

$$\int_0^1 \zeta'(\sigma_1 + it,\alpha)\zeta'(\sigma_2 - it,\alpha)\,d\alpha \tag{1}$$

的渐近性质, 这里 $0 < \sigma_1, \sigma_2 < 1$, $t \geq 2$.

关于这一问题, [1] 曾作过研究, 并利用 Hurwitz Zeta-函数的导数 $\zeta_1'(s,\alpha)$ 的逼近方程及其三角和估计证明了下面的结果:

$$\int_0^1 \left|\zeta'\left(\frac{1}{2} + it,\alpha\right)\right|^2 d\alpha = \frac{1}{3}\ln^3\left(\frac{t}{2\pi}\right) + \gamma\ln^2\left(\frac{t}{2\pi}\right) + 2\gamma_1\ln\left(\frac{t}{2\pi}\right) + \gamma_2 + O\left(t^{-\frac{1}{6}}(\ln t)^{\frac{10}{3}}\right),$$

其中 $\gamma_i = \lim_{N\to\infty}\left(\sum_{n\leq N}\frac{\ln^i n}{n} - \frac{1}{i+1}(\ln N)^{i+1}\right)$, $\gamma_0 = \gamma$ 为 Euler 常数.

本文对此作进一步的研究, 利用解析方法给出 (1) 式的一个很强的渐近公式.

引理　对于给定的 $0 < \sigma_1, \sigma_2 < 1$, 当 $t \geq 2$ 时有估计式

$$\int_0^1 \alpha^{-\sigma_1 + it}\ln\alpha\,\zeta'(\sigma_2 + it,\alpha)\,d\alpha = O\left(\frac{1}{t}\right).$$

证明　设 $s_1 = -\sigma_1 + it$, $s_2 = \sigma_2 + it$, 反复使用分部积分法得

$$\int_0^1 \alpha^{s_1}\ln\alpha\,\zeta'(s_2,\alpha)\,d\alpha$$

$$= \left[\frac{1}{1+s_1}\int_0^1 \alpha^{s_1}\zeta'(s_2,\alpha)\,d\alpha - \sum_{k=1}^{n}\frac{s_2(1+s_2)\cdots(k-1+s_2)}{(1+s_1)(2+s_1)\cdots(k+1+s_1)}\int_0^1 \alpha^{k+s_1}\zeta'(k+s_2,\alpha)\,d\alpha\right]$$

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$$-\left[\frac{1}{1+s_1}\int_0^1 \alpha^{1+s_1}\ln\alpha\,\zeta(1+s_2,\alpha)\,d\alpha+\sum_{k=1}^{n}\frac{s_2(1+s_2)\cdots(k-1+s_2)}{(1+s_1)(2+s_1)\cdots(k+1+s_1)}\cdot\int_0^1\alpha^{k+1+s_1}\ln\alpha\,\zeta(k+1+s_2,\alpha)\,d\alpha\right]$$

$$+\frac{s_2(1+s_2)\cdots(n+s_2)}{(1+s_1)(2+s_1)\cdots(n+1+s_1)}\int_0^1\alpha^{n+1+s_1}\ln\alpha\,\zeta'(n+1+s_2,\alpha)\,d\alpha\equiv J_1+J_2+J_3. \qquad (2)$$

现在分别估计(2)**中的各项 首先, 对** J_1 **中的第一项, 我们再使用分部积分法有**

$$-\frac{1}{1+s_1}\int_0^1\alpha^{s_1}\zeta'(s_2,\alpha)\,d\alpha$$

$$=-\frac{1}{1+s_1}\left[\frac{\zeta'(s_2,1)}{1+s_1}-\frac{1}{1+s_1}\int_0^1\alpha^{s_1+1}\zeta'(s_2+1,\alpha)\,d\alpha+\frac{s_2}{1+s_1}\int_0^1\zeta'(s_2+1,\alpha)\alpha^{s_1+1}\,d\alpha\right]\ll\frac{1}{t}, \qquad (3)$$

同理对 J_2 **中第一项有估计式**

$$-\frac{1}{1+s_1}\int_0^1\alpha^{1+s_1}\ln\alpha\,\zeta(1+s_2,\alpha)\,d\alpha\ll\frac{1}{t}, \qquad (4)$$

其次对于 J_1 **中的求和式, 有**

$$-\sum_{k=1}^{n}\frac{s_2(1+s_2)\cdots(k-1+s_2)}{(1+s_1)(2+s_1)\cdots(k+1+s_1)}\int_0^1\alpha^{k+s_1}\zeta'(k+s_2,\alpha)\,d\alpha$$

$$\ll\frac{1}{t}\sum_{k=1}^{n}\left[\int_0^{1/2}\alpha^{k-\sigma_1}\zeta'(k+\sigma_2,\alpha)\,d\alpha+\int_{1/2}^1\alpha^{k-\sigma_1}\zeta'(k+\sigma_2,\alpha)\,d\alpha\right]$$

$$\ll\frac{1}{t}\sum_{k=1}^{n}\left[\frac{1}{2^{k-\sigma_1}}+\int_{1/2}^1\zeta'(k+\sigma_2,\frac{1}{2})\,d\alpha\right]\ll\frac{1}{t}\sum_{k=1}^{n}\left[\frac{1}{2^{k-\sigma_1}}+\frac{1}{(\frac{3}{2})^{k+\sigma_2}}\right]\ll\frac{1}{t}. \qquad (5)$$

对 J_2 **中的求和式, 注意到** $0\leq\alpha<1$ **时** $|\alpha\ln\alpha|<1$, **于是有**

$$-\sum_{k=1}^{n}\frac{s_2(1+s_2)\cdots(k-1+s_2)}{(1+s_1)(2+s_1)\cdots(k+1+s_1)}\int_0^1\alpha^{k+1+s_1}\ln\alpha\,\zeta(k+1+s_2,\alpha)\,d\alpha$$

$$\ll\frac{1}{t}\sum_{k=1}^{n}\left[\int_0^{1/2}\alpha^{k-\sigma_1}\,d\alpha+\int_{1/2}^1\zeta(k+1+\sigma_2,\frac{1}{2})\,d\alpha\right]$$

$$\ll\frac{1}{t}\sum_{k=1}^{n}\left[\frac{1}{2^{k-\sigma_1}}+\frac{1}{(\frac{3}{2})^{k+1}}\right]\ll\frac{1}{t}. \qquad (6)$$

最后估计 J_3, **取** $n\geq 3\ln t$, **就有**

$$J_3\ll\int_0^1\alpha^{n-\sigma_1}|\alpha\ln\alpha|\,|\zeta'(n+1+\sigma_2,\alpha)|\,d\alpha\ll\int_0^{1/2}\alpha^{n-\sigma_1}\,d\alpha+\int_{1/2}^1|\zeta'(n+1+\sigma_2,\alpha)|\,d\alpha$$

$$\ll\frac{1}{2^{n-\sigma_1}}+\frac{1}{(\frac{3}{2})^{n+1}}]\ll\frac{1}{t}. \qquad (7)$$

由(2), (3), (4), (5), (6)**及**(7)**立刻得到** $J_1+J_2+J_3\ll\frac{1}{t}$, **于是完成了引理的证明**

§2　主要结果

这节给出本文的主要结果

定理 1　对于给定的 $0<\sigma_1,\sigma_2<1$，且 $\sigma_1+\sigma_2\neq 1$，则当 $t\geq 2$ 时有

$$\int \zeta'(\sigma_1+it,\alpha)\zeta'(\sigma_2-it,\alpha)\,d\alpha=\frac{2}{(\sigma_1+\sigma_2+1)^3}+\left(\frac{t}{2\pi}\right)^{1-\sigma_1-\sigma_2}\zeta''(2-\sigma_1-\sigma_2)$$

$$+2\left(\frac{t}{2\pi}\right)^{1-\sigma_1-\sigma_2}\ln\left(\frac{t}{2\pi}\right)\zeta'(2-\sigma_1-\sigma_2)+\left(\frac{t}{2\pi}\right)^{1-\sigma_1-\sigma_2}\ln^2\left(\frac{t}{2\pi}\right)\zeta(2-\sigma_1-\sigma_2)+O\left(\frac{\ln^2 t}{t^{\sigma_1+\sigma_2}}\right)$$

$$+O\left(\frac{1}{t}\right),$$

其中 $\zeta(s)$ 为 Riemann Zeta-函数

证明　首先当 $\mathrm{Re}(s)>1$ 时，由 Hurwitz Zeta-函数的函数方程知

$$\zeta(1-s,a)=\frac{\Gamma(s)}{(2\pi)^s}\{e^{-\frac{\pi is}{2}}F(a,s)+e^{\frac{\pi is}{2}}F(-a,s)\},$$

其中 $F(a,s)=\sum_{n=1}^{\infty}\frac{e^{2\pi ina}}{n^s}$，$0<a\leq 1$. 对上式两端 s 求导可得

$$-\zeta'(1-s,a)=\frac{\Gamma(s)}{(2\pi)^s}\left(\frac{\Gamma'(s)}{\Gamma(s)}-\ln(2\pi)\right)\left(e^{-\frac{\pi is}{2}}F(a,s)+e^{\frac{\pi is}{2}}F(-a,s)\right)$$

$$+\frac{\Gamma(s)}{(2\pi)^s}\{e^{-\frac{\pi is}{2}}\frac{\partial}{\partial s}F(a,s)-\frac{\pi i}{2}\left(e^{-\frac{\pi is}{2}}F(a,s)-e^{\frac{\pi is}{2}}F(-a,s)\right)+e^{\frac{\pi is}{2}}\frac{\partial}{\partial s}F(-a,s)\},\tag{8}$$

其中 $\frac{\partial}{\partial s}F(\pm s,a)=-\sum_{n=1}^{\infty}\frac{e^{\pm 2\pi ina}\ln n}{n^s}$. 于是对 $0<\sigma_1,\sigma_2<1$，考虑函数

$$A(w,\sigma_1,\sigma_2,t)=\int \zeta'(\sigma_1+it+w,\alpha)\zeta'(\sigma_2-it+w,\alpha)\,d\alpha$$

当 $\mathrm{Re}(w)\leq -1$ 时，$\sigma_i+\mathrm{Re}(w)<0$，于是由(8)并注意三角积分

$$\int e^{2\pi in\alpha}\,d\alpha=\begin{cases}1, & n=0,\\ 0, & n\neq 0,\end{cases}$$

以及设 $\tau_1=1-\sigma_1-it-w$，$\tau_2=1-\sigma_2+it-w$，$\tau=2-\sigma_1-\sigma_2-2w$ 可以得到

$$A(w,\sigma_1,\sigma_2,t)$$

$$=\frac{2\Gamma(\tau_1)\Gamma(\tau_2)}{(2\pi)^{\tau}}-\cos\left(\pi\left(\frac{\sigma_2-\sigma_1}{2}-it\right)\right)\{\left(\frac{\Gamma'(\tau_1)}{\Gamma(\tau_1)}+\frac{\pi i}{2}-\ln(2\pi)\right)\left(\frac{\Gamma'(\tau_2)}{\Gamma(\tau_2)}-\frac{\pi i}{2}-\ln(2\pi)\right)\zeta(\tau)$$

$$+\left(\frac{\Gamma'(\tau_1)}{\Gamma(\tau_1)}+\frac{\Gamma'(\tau_2)}{\Gamma(\tau_2)}-2\ln(2\pi)\right)\zeta'(\tau)+\zeta''(\tau)$$

$$+\frac{\pi i}{2}\left(\frac{\Gamma'(\tau_2)}{\Gamma(\tau_2)}-\frac{\Gamma'(\tau_1)}{\Gamma(\tau_1)}\right)\left(\frac{i\sin\left(\pi\left(\frac{\sigma_2-\sigma_1}{2}-it\right)\right)}{\cos\left(\pi\left(\frac{\sigma_2-\sigma_1}{2}-it\right)\right)}-1\right)\zeta(\tau)\}.\tag{9}$$

另一方面，当 $\mathrm{Re}(w)<-1$ 时有

$$A(w,\sigma_1,\sigma_2,t)=\int \zeta_1'(\sigma_1+it+w,\alpha)\zeta_1'(\sigma_2-it+w,\alpha)\,d\alpha$$

$$-\int \alpha^{-(\sigma_1+it+w)}\ln\alpha\,\zeta_1'(\sigma_2-it+w,\alpha)\,d\alpha$$

$$-\int\alpha^{-(\sigma_2+it+w)}\ln\alpha\,\zeta'(\sigma_1+it+w,\alpha)\,d\alpha+\frac{2}{(\tau-1)^3}. \tag{10}$$

结合(9)及(10)式可得

$$\int\zeta'(\sigma_1+it+w,\alpha)\zeta'(\sigma_2-it+w,\alpha)\,d\alpha$$

$$=\frac{2}{(1-\tau)^3}+\int\alpha^{-(\sigma_1+it+w)}\ln\alpha\,\zeta'(\sigma_2-it+w,\alpha)\,d\alpha+\int\alpha^{-(\sigma_2-it+w)}\ln\alpha\,\zeta'(\sigma_2-it+w,\alpha)\,d\alpha$$

$$+\frac{\Gamma(\tau_1)\Gamma(\tau_2)2\cos(\pi(\frac{\sigma_2-\sigma_1}{2}-it))}{(2\pi)^{\tau}}\{(\frac{\Gamma'(\tau_1)}{\Gamma(\tau_1)}+\frac{\pi i}{2}-\ln(2\pi))(\frac{\Gamma'(\tau_2)}{\Gamma(\tau_2)}-\frac{\pi i}{2}-\ln(2\pi))\zeta(\tau)$$

$$+(\frac{\Gamma'(\tau_1)}{\Gamma(\tau_1)}+\frac{\Gamma'(\tau_2)}{\Gamma(\tau_2)}-2\ln(2\pi))\zeta'(\tau)+\zeta''(\tau)$$

$$+\frac{\pi i}{2}(\frac{\Gamma'(\tau_2)}{\Gamma(\tau_2)}-\frac{\Gamma'(\tau_1)}{\Gamma(\tau_1)})(\frac{i\sin(\pi(\frac{\sigma_2-\sigma_1}{2}-it))}{\cos(\pi(\frac{\sigma_2-\sigma_1}{2}-it))}-1)\zeta(\tau)\}. \tag{11}$$

由解析开拓可知上式对所有 $\mathrm{Re}(w)\le 0$ 成立

对给定的 $0<\sigma_1,\sigma_2<1$, 由 Stirling 公式[4]不难推出

$$\frac{\Gamma(1-\sigma_1-it)\Gamma(1-\sigma_2+it)}{(2\pi)^{1-\sigma_1-\sigma_2}}\times 2\cos(\pi(\frac{\sigma_2-\sigma_1}{2}-it))=(\frac{t}{2\pi})^{1-\sigma_1-\sigma_2}\{1+O(\frac{1}{t})\}, \tag{12}$$

当 $\sigma_1+\sigma_2\ne 1$ 时, 在(11)中取 $w=0$, 应用[1]中引理 3, 并由(12)式及引理立即得到定理 1.

定理 2　对于给定的 $0<\sigma<1$, 则当 $t\ge 2$ 时有

$$\int\zeta'(\sigma+it,\alpha)\zeta'(1-\sigma-it,\alpha)\,d\alpha=\frac{1}{3}\ln^3(\frac{t}{2\pi})+\gamma\ln^2(\frac{t}{2\pi})+2\gamma_1\ln(\frac{t}{2\pi})+\gamma_2+O(\frac{\ln^2 t}{t}).$$

证明　当 $\sigma_1+\sigma_2=1$ 时, 令 $\sigma_1=\sigma$, 则 $\sigma_2=1-\sigma$, 于是在(11)中取 $w\to 0$ 并应用引理可得

$$\int\zeta'(\sigma+it,\alpha)\zeta'(1-\sigma-it,\alpha)\,d\alpha$$

$$=\int\alpha^{-(\sigma+it)}\ln\alpha\,\zeta'(1-\sigma-it,\alpha)\,d\alpha+\int\alpha^{-(1-\sigma-it)}\ln\alpha\,\zeta'(\sigma+it,\alpha)\,d\alpha$$

$$+\lim_{w\to 0}\{\frac{\Gamma(1-\sigma-it-w)\Gamma(1+it-w)2\sin(\pi(\sigma+it))}{(2\pi)^{1-2w}}[(\frac{\Gamma'(1-\sigma-it-w)}{\Gamma(1-\sigma-it-w)}$$

$$+\frac{\pi i}{2}-\ln(2\pi))(\frac{\Gamma'(\sigma+it-w)}{\Gamma(\sigma+it-w)}-\frac{\pi i}{2}-\ln(2\pi))\zeta(1-2w)$$

$$+(\frac{\Gamma'(1-\sigma-it-w)}{\Gamma(1-\sigma-it-w)}+\frac{\Gamma'(\sigma+it-w)}{\Gamma(\sigma+it-w)}-2\ln(2\pi))\zeta'(1-2w)+\zeta''(1-2w)$$

$$+\frac{\pi i}{2}(\frac{\Gamma'(1+it-w)}{\Gamma(1+it-w)}-\frac{\Gamma'(1-\sigma-it-w)}{\Gamma(1-\sigma-it-w)})(\frac{i\cos(\pi(\sigma+it))}{\sin(\pi(\sigma+it))}-1)\zeta(1-2w)]+\frac{1}{4w^3}\}$$

$$=\gamma_2+2\gamma_1\ln(\frac{t}{2\pi})+\gamma\ln^2(\frac{t}{2\pi})-A(\sigma,t)+O(\frac{1}{t}). \tag{13}$$

经简单计算得

$$A(\sigma,t)=-\frac{1}{3}\ln^3(\frac{t}{2\pi})+O(t^{-1}\ln^2 t), \tag{14}$$

其中用到[5]

$$(\frac{\Gamma'(\sigma\pm it)}{\Gamma(\sigma\pm it)})'=O(t^{-1}\ln^2 t),\quad(\frac{\Gamma'(\sigma\pm it)}{\Gamma(\sigma\pm it)})''=O(t^{-1}\ln^2 t),$$

$$\frac{\Gamma'(1-\sigma-it)}{\Gamma(1-\sigma-it)}-\frac{\Gamma'(\sigma+it)}{\Gamma(\sigma+it)}=\frac{\pi\cos(\pi(\sigma+it))}{\sin(\pi(\sigma+it))}.$$

由(13)及(14)式,即得定理 2

由定理 1 及定理 2 立得下面的

推论 1 当 $t\geq 2$ 时有渐近公式

$$\int|\zeta'(\frac{1}{2}+it,\alpha)|^2 d\alpha=\frac{1}{3}\ln^3(\frac{t}{2\pi})+\gamma\ln^2(\frac{t}{2\pi})+2\gamma_1\ln(\frac{t}{2\pi})+\gamma_2+O(\frac{\ln^2 t}{t}).$$

推论 2 当 $t\geq 2$, $0<\sigma<1$, 且 $\sigma\neq\frac{1}{2}$ 时有

$$\int|\zeta'(\sigma+it,\alpha)|^2 d\alpha=\frac{2}{(2\sigma-1)^3}+(\frac{t}{2\pi})^{1-2\sigma}\zeta''(2-2\sigma)+2(\frac{t}{2\pi})^{1-2\sigma}\ln(\frac{t}{2\pi})\zeta'(2-2\sigma)$$
$$+(\frac{t}{2\pi})^{1-2\sigma}\ln^2(\frac{t}{2\pi})\zeta(2-2\sigma)+O(\frac{\ln^2 t}{t^{2\sigma}})+O(\frac{1}{t}).$$

推论 3 设 $0<\sigma_1,\sigma_2<1$, 且 $\sigma_1+\sigma_2>1$, 则

$$\lim_{|t|\to\infty}\int|\zeta'(\sigma_1+it,\alpha)\zeta'(\sigma_2-it,\alpha)|d\alpha=\frac{2}{(\sigma_1+\sigma_2-1)^3}.$$

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On the Mean Value Formula of the Derivative of Hurwitz Zeta-Function

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Abstract

We give a sharper asymptotic formula for the integral mean value of $\zeta'(s,\alpha)$ for parameter α.

Keywords Hurwitz Zeta-function, mean value, asymptotic formula.