Remark From the proof of Theorem 4 one can see that for even n,

$$n\left(\frac{d_n}{(n-1)!}-1\right)=2+O(n^{-1}).$$

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关于Witt向量之序列

谢廷藩　　　　周颂平

(中国计量学院, 杭州 310034)　(杭州大学数学系, 310027)

摘　要

对出现于公式 $\prod_{n\geq 1}\frac{1}{1+d_n t^{n/n!}}=(1-t)e^t$ **中的整数** d_n **给出了精确估计.**

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On a Sepuence of Witt Vectors*

Xie Tingfan
(China Institute of Mettrology, Hangzhou 310034)
Zhou Songping
(Dept. of Math., Hangzhou University, 310027)

Abstract A sharp estimate for the integers d_n, in the formula $\Pi_{n\geq 1}1/(1+d_nt^n/n!)=(1-t)e^t$ is obtained.

Keywords sequence, witt vectors
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1. Introduction

Let A be a commutative ring, $W(A)$ be the ring of Witt vectors over A, and let $\Lambda(A)$ be the free λ- ring.

It is know from [2] that $(q_n)_{n\geq 1}\to\Pi_{n\geq 1}(1-q_nt^n)^{-1}$ defines an isomorphism between $W(A)$ and $\Lambda(A)$.

It is clear that q_n in $\Pi_{n\geq 1}\frac{1}{1-q_nt^n}=\sum_{n\geq 0}h_nt^n$ correspond to representations of the nth symmetric group via the characteristic map. The character table of these representations gives formulae expressing the components of a Witt vector as a function of its "ghost components" (cf. [3, p. 352]).

Now write

$$\Pi_{n\geq 1}\frac{1}{1+d_nt^n/n!}=(1-t)e^t. \tag{1}$$

Since sequence $\{d_n\}$ in (1) gives the dimensions of these representations, it is of interest to investigate its asymptotic behavior. Recently, Borwein and Lou[1] proved that

Theorem BL For $n=2,3,\cdots$,

$$d_n\leq(n-1)!\quad\text{if } n \text{ is an odd number;}\tag{2}$$
$$d_n\leq(n-1)!\quad\text{if } n \text{ is a prime;}\tag{3}$$
$$d_n\geq(n-1)!\quad\text{if } n \text{ is an even number.}\tag{4}$$

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Furthermore,

$$1 - \frac{1}{n} \leq \frac{d_n}{(n-1)!} \leq 1 + \frac{\alpha_n}{\sqrt{n}}, \tag{5}$$

where $\alpha_8 = \alpha_{16} = 2$, and other $\alpha_n = 1$.

We notice that the inequality (5) is not sharp since, for example,

$$\frac{d_8}{7!} = \frac{8505}{5040} < 1 + \frac{2}{\sqrt{8}}.$$

So it is natural to look for the sharp estimates for $d_n/(n-1)!$.

The present paper will answer this question by the following theorems, in an approach different from the others by using some new ideas in approximation theory.

Theorem 1 For $n = 2k+1$, $k = 1, 2, \cdots,$, we have

$$0 \leq 1 - \frac{d_n}{(n-1)!} \leq \frac{1}{n^2}\frac{106}{9}. \tag{6}$$

Furthermore,

$$\max_{n\ \mathrm{odd}} n^2\left(1 - \frac{d_n}{(n-1)!}\right) = 15^2\left(1 - \frac{d_{15}}{14!}\right) = \frac{106}{9}, \tag{7}$$

and $n = 15$ is the only number where the maximum is attained.

Theorem 2 For $n = 2k$, $k = 1, 2, \cdots,$, we have

$$0 \leq \frac{d_n}{(n-1)!} - 1 \leq \frac{1}{n}\frac{907}{128}. \tag{8}$$

Furthermore,

$$\max_{n\ \mathrm{even}} n\left(\frac{d_n}{(n-1)!} - 1\right) = 16\left(\frac{d_{16}}{15!} - 1\right) = \frac{907}{128}, \tag{9}$$

and $n = 16$ is the only number where the maximum is attained.

Theorem 3 For $n = 2k+1$, $k = 1, 2, \cdots,$, we have

$$\limsup_{n\to\infty} n^2\left(1 - \frac{d_n}{(n-1)!}\right) = 9. \tag{10}$$

Theorem 4 For $n = 2k$, $k = 1, 2, \cdots,$, we have

$$\lim_{n\to\infty} n\left(\frac{d_n}{(n-1)!} - 1\right) = 2 \tag{11}$$

2 Proof of Theorems 1 and 3

In [1], J. Borwein and S. T. Lou have proved that for natural numbers $n > 1$ one has

$$\frac{d_n}{(n-1)!} = 1 + n \sum_{\substack{kh=n \\ k \neq 1, n}} \frac{(-1)^h}{h} \left(\frac{d_k}{k!} \right)^h, \tag{12}$$

from which it follows that

$$\frac{d_2}{1!} = 1, \frac{d_3}{2!} = 1, \frac{d_4}{3!} = 1 + \frac{1}{2}, \frac{d_5}{4!} = 1, \frac{d_6}{5!} = 1 + \frac{1}{12}, \frac{d_7}{6!} = 1,$$

$$\frac{d_8}{7!} = 1 + \frac{11}{16}, \frac{d_9}{8!} = 1 - \frac{1}{9}, \frac{d_{10}}{9!} = 1 + \frac{11}{80}, \frac{d_{11}}{10!} = 1, \frac{d_{12}}{11!} = 1 + \frac{183}{3456},$$

$$\frac{d_{13}}{12!} = 1, \frac{d_{14}}{13!} = 1 + \frac{57}{448}, \frac{d_{15}}{14!} = 1 - \frac{106}{2025}, \frac{d_{16}}{15!} = 1 + \frac{907}{2048}, \frac{d_{17}}{16!} = 1, \cdots.$$

Proof of Theorem 1 In this section we always assume that n are odd numbers. Define $g(h, k) =: h^2k^3 \left(\frac{d_k}{k!} \right)^h$. Since each n is odd, we see that h and k are odd. From (12), we get

$$\begin{aligned} n^2 \left(1 - \frac{d_n}{(n-1)!} \right) &= \sum_{\substack{kh=n \\ k \neq 1, n}} g(h, k) = \sum_{\substack{kh=n \\ k \neq 1, n \\ k=3}} g(h, k) + \sum_{\substack{kh=n \\ k \neq 1, n \\ k=5}} g(h, k) \\ &\quad + \sum_{\substack{kh=n \\ k \neq 1, n \\ k, h \geq 7}} g(h, k) + \sum_{\substack{kh=n \\ h=3 \text{ or } h=5 \\ k \neq 1, n}} g(h, k) \\ &\leq g(h, 3) + g(h, 5) + \sum_{\substack{kh=n \\ k, h \geq 7}} g(h, k) + \sum_{\substack{kh=n \\ k \neq 1, n \\ h=3 \text{ or } h=5}} g(h, k). \end{aligned} \tag{13}$$

We will show that the sum of the first three terms in this series is not bigger than 1. First we show that

$$\sum_{\substack{kh=n \\ k \neq 1, n \\ k, h \geq 7}} h^2k^3 \left(\frac{d_k}{k!} \right)^h < \frac{1}{6}. \tag{14}$$

Since $\sum_{\substack{kh=n \\ k, h \geq 7}} \frac{1}{k^2} \leq \sum_{k=7}^{\infty} \frac{1}{k^2} \leq \frac{1}{6}$, we only need to prove that

$$h^2k^3 \left(\frac{d_k}{k!} \right)^h \leq \frac{1}{k^2}. \tag{15}$$

By Theorem BL, $\frac{d_k}{k!} \le \frac{1}{k}$, so what we have to prove is only that $h^2 k^{5-h} \le 1$, *or*

$$f(k, h) := 2\log h + (5 - h)\log k \le 0. \tag{16}$$

It is evident that in the case $k \ge 7$, (16) holds for $h = 7$, while $\frac{d}{dh} f(k, h) = \frac{2}{h} - \log k < 0$ holds for $h > 7$, that is, $f(k, h)$ is a decreasing function with respect to h, hence (16) holds, and consequently (15) as well as (14) holds

Now we investigate

$$g(h, 3) = 27h^2 \left(\frac{d_3}{3!}\right)^h = \frac{27h^2}{3^h}.$$

It is not difficult to see that for $h \ge 7$, $g(h, 3)$ is decreasing and $g(7, 3) = 49/81$, hence

$$g(h, 3) \le \frac{49}{81} \quad (h \ge 7), \tag{17}$$

similarly,

$$g(h, 5) = 125h^2 \left(\frac{d_5}{5!}\right)^h \le \frac{49}{625} \quad (h \ge 7) \tag{17'}$$

Now combining (13), (14), (17) and (17') we have

$$g(h, 3) + g(h, 5) + \sum_{\substack{kh = n \\ k, h \ge 7}} g(h, k) \le \frac{49}{81} + \frac{49}{625} + \frac{1}{6} \le 1.$$

Write

$$\sum_{\substack{hk = n,\ h = 3,\ h = 5 \\ k \ne 1, n}} h^2 k^3 \left(\frac{d_k}{h!}\right) =: I_n. \tag{18}$$

To estimate I_n, we consider the following cases

(i) If n does not have factors 3 and 5, then $I_n = 0$;

(ii) If $n > 15$ can be divided by 15, then from (2),

$$I_n \le \frac{n^3}{3}\left(\frac{d_{n/3}}{(n/3)!}\right)^3 + \frac{n^3}{5}\left(\frac{d_{n/5}}{(n/5)!}\right)^5 = 9 + \frac{5^4}{n^2} \le 9 + \frac{25}{81} < 9 + \frac{25}{9} - 1;$$

(iii) If n is divisible by 3 but not by 5, then $I_n \le 9$;

(iv) If n is divisible by 5 but not by 3, then

$$I_n \le \frac{25}{49};$$

(v) If $n = 15$, then from (3),

$$n^2 \left(1 - \frac{d_n}{(n-1)!}\right) = 9 + \frac{25}{9}.$$

All these estimates together with (18) imply that

$$n^2\left(1-\frac{d_n}{(n-1)!}\right)\le 9+\frac{25}{9}=15^2\left(1-\frac{d_{15}}{14!}\right).$$

Theorem 1 is completed. □

Proof of Theorem 3 Due to the same reason, since each n is odd, in the following k, h must be odd. From the proof of Theorem 1 one can see that if $n>15$, then

$$n^2\left(1-\frac{d_n}{(n-1)!}\right)\le 9+\frac{5^4}{n^2}+\sum_{\substack{hk=n\\ k\ne 1,n\\ k,h\ge 7}}h^2k^{3-h},$$

furthermore, for any given natural number $p>7$ one has

$$n^2\left(1-\frac{d_n}{(n-1)!}\right)\le 9+\frac{5^4}{n^2}+\sum_{\substack{7\le k\le p\\ kh=n}}h^2k^{3-h}+\sum_{k>p}\frac{1}{k^2}.$$

Since in the case $7\le k\le p$, $h^2k^{3-h}\to 0$ as $n\to\infty$, we have

$$\limsup_{n\to\infty}n^2\left(1-\frac{d_n}{(n-1)!}\right)\le 9+\frac{1}{p},$$

or

$$\limsup_{n\to\infty}n^2\left(1-\frac{d_n}{(n-1)!}\right)\le 9 \tag{19}$$

in view of that p is any given natural number not less than 7. On the other hand, take

$$n=3q_k,$$

where q_k is the kth prime, then

$$n^2\left(1-\frac{d_n}{(n-1)!}\right)=\frac{n^3}{3}\left(\frac{d_{n/3}}{(n/3)!}\right)^3+\frac{n^3}{q_k}\left(\frac{d_3}{3!}\right)^{q_k}=9+\frac{n^3}{q_k}\left(\frac{1}{3}\right)^{q_k}=9+\frac{27q_k^2}{3^{q_k}}.$$

Therefore

$$\lim_{n\to\infty}9q_k^2\left(1-\frac{d_{3q_k}}{(3q_k-1)!}\right)=9. \tag{20}$$

(19) and (20) together yields (10), which is the required equality. □

Remark We see that if n does not have the factor 3, then $n^2(1-d_n/(n-1)!)$ will be much less than 9. In fact we have then

$$1-\frac{d_n}{(n-1)!}=O\left(n^{-4}\right).$$

Starting from it, we can establish a corresponding result for $\{d_{n_j}\}$ for some particular subsequence $\{n_j\}$.

3. Proof of Theorems 2 and 4

For notational convenience, we assume n be even numbers in this section.

Proof of Theorem 2 From (12),

$$n\left(\frac{d_n}{(n-1)!}-1\right) \sum_{\substack{hk=n\\ k\neq 1,n\\ h \text{ even}}} k^2 h\left(\frac{d_k}{k!}\right)^h = \sum_{\substack{hk=n\\ k\neq 1,n\\ h \text{ even},\ h\geq 8}} k^2 h\left(\frac{d_k}{k!}\right)^h$$

$$+ \sum_{\substack{hk=n\\ k\neq 1,n\\ h \text{ even},\ h\geq 6}} k^2 h\left(\frac{d_k}{k!}\right)^h := I_1 + I_2. \tag{21}$$

Write

$$f(k,h) = \log\left(lk^4 h\left(\frac{d_k}{k!}\right)^h\right)$$

In a similar way to the proof of Theorem 1, by applying Theorem BL, from $d_k/(k-1)! \leq 2$ we have

$$f(k,h) \leq 0$$

for $k \geq 8$ and $h \geq 8$, hence

$$k^2\left(\frac{d_k}{k!}\right)^h \leq \frac{1}{k^2} \quad (k \geq 8,\ h \geq 8). \tag{22}$$

By using again the known results

$$\frac{d_2}{1!} = 1,\ \frac{d_4}{3!} = \frac{3}{2},\ \frac{d_6}{5!} = \frac{13}{12},$$

we can prove that for $h \geq 8$,

$$k^2\left(\frac{d_k}{k!}\right)^h \leq \frac{1}{k^2} \quad (k = 2, 4, 6),$$

together with (22) we obtain that

$$I_1 \leq \sum_{k=2}^{\infty} \frac{1}{k^2} \leq \frac{3}{4} \quad (h \geq 8). \tag{23}$$

From the estimate of Theorem BL $\frac{d_n}{(n-1)!} \leq 1 + \frac{a_n}{\sqrt{n}}$, direct calculation will lead to that

$$\frac{n^2}{2}\left(\frac{d_{n/2}}{(n/2)!}\right)^2 + \frac{n^2}{4}\left(\frac{d_{n/4}}{(n/4)!}\right)^4 + \frac{n^2}{6}\left(\frac{d_{n/6}}{(n/6)!}\right)^6 < 5$$

for $n \geq 36$, and

$$I_2 = \sum_{\substack{kh = n \\ k \neq 1, n \\ n \text{ even}, h \leq 6}} \frac{n^2}{h(n/h)^h} \left(\frac{d_k}{(k-1)!} \right)^h$$

$$\leq 2\left(1 + \frac{2}{\sqrt{n}}\right)^2 + \frac{4^3}{n^2}\left(1 + \frac{2}{\sqrt{n}}\right)^4 + \frac{6^5}{n^4}\left(1 + \frac{2}{\sqrt{n}}\right)^6 \leq 6.1$$

if $n \geq 18$. We thus have by (21) and (23) that $n\left(\frac{d_n}{(n-1)!} - 1\right) \leq 7 \quad (n \geq 18)$.

For $n \leq 18$, we directly calculate that

$$\max_{\substack{2 \leq n \leq 18 \\ n \text{ even}}} n\left(\frac{d_n}{(n-1)!} - 1\right) = 16\left(\frac{d_{16}}{15!} - 1\right) = \frac{907}{128}.$$

Therefore, we finally obtain that

$$n\left(\frac{d_n}{(n-1)!} - 1\right) = 16\left(\frac{d_{16}}{15!} - 1\right) = \frac{907}{128}.$$

Theorem 2 is proved. □

Proof of Theorem 4 From (12) together with the fact that n is even,

$$\left| n\left(\frac{d_n}{(n-1)!} - 1\right) - \frac{n^2}{2}\left(\frac{d_{n/2}}{(n/2)!}\right)^2 \right| \leq \sum_{i=3}^{6} \Delta_i(n) + \sum_{\substack{hk = n, k \neq 1, n \\ h \geq 8 \\ k \leq s}} k^2 h \left(\frac{d_k}{k!}\right)^h + \sum_{\substack{hk = n, k \neq 1, n \\ h \geq 8 \\ k > s}} k^2 h \left(\frac{d_k}{k!}\right)^h, \tag{24}$$

where

$$\Delta_i(n) = \begin{cases} \frac{n^2}{i}\left(\frac{d_{n/i}}{(n/i)!}\right)^i, & \text{if is a factor of } n, \\ 0, & \text{otherwise.} \end{cases}$$

It is not difficult to see that $\lim_{m \to \infty} \frac{d_m}{(m-1)!} = 1$ implying that $\lim_{n \to \infty} \Delta_i(n) = 0$, then on the basis of the estimate $k^2 h\left(\frac{d_k}{k!}\right)^h \leq \frac{1}{k^2}$ $(h \geq 8,\ k \geq 8)$ in the proof of Theorem 2, we deduce from (24) that

$$\limsup_{n \to \infty} \left| n\left(\frac{d_n}{(n-1)!} - 1\right) - 2\left(\frac{d_{n/2}}{(n/2-1)!}\right)^2 \right| \leq \sum_{k >} \frac{1}{k^2},$$

that is, $\lim_{n \to \infty} \left| n\left(\frac{d_n}{(n-1)!} - 1\right) - 2 \right| = 0$, since s is arbitrarily given and $\lim_{n \to \infty} \left(\frac{d_{n/2}}{(n/2-1)!}\right) = 1$, therefore we have $\lim_{n \to \infty} n\left(\frac{d_n}{(n-1)!} - 1\right) = 2$, which completes Theorem 4. □