Proof For any root subgroup $U_a(Q)$, by Lemma 3.3 $\pi(U_a(Q))$ is a unipotent subgroup of SL (V), hence $\pi(U_a(Q))$ is conjugate in SL (V) to a subgroup of the unipotent upper triangular matrix group. Since the expotential map Exp of nilpotent upper triangular matrices to unipotent upper triangular matrices is bijective, for any i there exist nilpotent linear transformations $d\pi(e_i)$ and $d\pi(f_i)$ on V such that

$$\pi(U_{a_i}(t)) = \exp td\pi(e_i), \quad \pi(U_{-a_i}(t)) = \exp td\pi(f_i)$$

for any $t \in Q$. By Lemma 3.1 and Theorem 3.2 the theorem follows □

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Kac-Moody 群上可微分模的刻画

查建国　黄临文

(上海同济大学应用数学系, 200092)

摘要

对 Kay-Moody 代数 g 上的任意可积模 $(V, d\pi)$, 通过指数可以把它提升为同 g 关联的 Kac-Moody 群 G 上的模 (V, π), G 上的这种模称为可微分模. 本文将刻画 G 上的可微分模并且证明, 模 (V, π) 是可微分模当且仅当 V 到每个根子群 U_a 的限制都是 U_a 的一个有理表示. 依据这种刻画, 得到一个有趣的结果: 有理数域 Q 上的 Chevalley 群 $G(Q)$ 的所有有限维模都是可微分模.

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A Characterization of Differentiable Modules over Kac- Moody Groups*

Zha Jianguo Huang Linwen
(Dept of Appl. Math., Tongji University, Shanghai 200092)

Abstract For any integrable module $(V, d\pi)$ over a Kac- Moody algebra g' we can lift it to a module (V, π) over the Kac- Moody group G associated to g'. The present paper characterizes such modules and shows that a G- module (V, π) is a such module lifted from an integrable g'- module (V, π) if and only if the restriction of V to any root subgroup U_a is rational. By the characterization, we obtain an interesting result which says that all finite dimensional modules over $G(Q)$ are differentiable for the Chevalley group $G(Q)$.

Keywords Kac- Moody groups, Chevalley groups, Modules
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1. Introduction

Throughout the paper the bass field F is of charcteristic zero.

The Kac- Moody algebra $g' = g'(A)$ associated to a symmetrizable generalized Cartan matrix $A = (a_{ij})_{n\times n}$ is the Lie algebra over F generated by 3n generators h_i, e_i and $f_i (1\le i\le n)$ with defining relations: [1]

1.1) $[h_i, h_j]= 0, \quad [e_i, f_j]= \delta_{ij}h_i;$

1.2) $[h_i, e_j]= a_j(h_i)e_j, \quad [h_i, f_j]= - a_j(h_i)f_j;$

1.3) $(ade_i)^{1- a_{ij}}(e_j)= 0, \quad (adf_i)^{1- a_{ij}}(f_j)= 0 \quad (i\neq j),$

where $a_j (1\le j\le n)$ is defined to be $a_j(h_i)= a_{ij}$. Corresponding to the Kac- Moody algebra g', we can construct the group $G= G(F)$, called the Kac- Moody group over F, associated to g' by representation approach[2]. We now recall the structure of the group G associated to g'. G is generated by $U_{\pm a_i}(t)$, $1\le i\le n$, $t\in F$, and the following fundamental relations hold[3].

R1) $U_a= \{U_a(t); t\in F\}$, called the root subgroup corresponding to the root a, is isomorphic to the additive group of the field F, where $a= a_i$ or $a= - a_i$.

R2) $(U_{a_i}, U_{-a_j})= 1$ for $i\neq j$.

R3) The subgroup G_{a_i} generated by $U_{\pm a_i}$ is isomorphic to $SL_2(F)$ and the isomorphism ϕ_i: $SL_2(F)\to G_{a_i}$ is given by

$$\phi_i\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix} = U_{a_i}(t), \quad \phi_i\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix} = U_{-a_i}(t).$$

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R4) Let $H_{\alpha_i}(t) = n_{\alpha_i}(t)n_{\alpha_i}(1)^{-1}$, where

$$n_{\alpha_i}(t) = U_{\alpha_i}(t)U_{-\alpha_i}(-t^{-1})U_{\alpha_i}(t).$$

The subgroup $H_{\alpha_i} = \{H_{\alpha_i}(t);\ t \in F^*\}$ is isomorphic to the multiplicative group F^* and the subgroup H generated by H_{α_i}, $1 \le i \le n$ is the directed product of the group H_{α_i}.

R5) $H_{\alpha_i}(t)U_{\pm\alpha_j}(u)H_{\alpha_i}(t)^{-1} = U_{\pm\alpha_j}(t^{\pm a_{ij}}u)$ for any $t \in F^*$, $u \in F$.

Definition 1.1 A module (V, π) over G is called differentiable if there exists an integrable action $d\pi$ of g' on V such that

$$\pi U_{\alpha_i}(t) = \exp th\pi e_i; \quad \pi U_{-\alpha_i}(t) = \exp th\pi f_i$$

for any $t \in F$.

Kac[4] conjectured that a G-module (V, π) is differentiable if the linear transformations on V

$$d\pi e_i = \frac{d}{dt}\pi U_{\alpha_i}(t)\Big|_{t=0}, \quad d\pi f_i = \frac{d}{dt}\pi U_{-\alpha_i}(t)\Big|_{t=0}.$$

can be defined. The present paper will justify the conjecture and show that a module (V, π) over G is differentiable if and only if the restriction of V to the subgroups $U_{\pm\alpha_i}$, $1 \le i \le n$ is a rational representation of $U_{\pm\alpha_i}$.

2 Integrable modules over the Kac-Moody algebra g′

Let h' denote the subspace spanned by $h_1, h_2, \cdots, h_n$. By Kac[5], an integrable module over g' is defined as follows:

Definition 2.1 A module $(V, d\pi)$ over g' is called integrable if the following conditions are satisfied:

i) $d\pi(h')$ is diagonalizable;

ii) all $d\pi(e_i)$ and $d\pi(f_i)$, $1 \le i \le n$, are locally nilpotent on V.

Proposition **2.2** A module $(V, d\pi)$ over g' is integrable only if the above condition ii) is satisfied.

Proof Since h' is an abelian Lie algebra, it will suffice to prove that all $d\pi(h_i)$, $1 \le i \le n$, is diagonalizable. Let g_i denote the subalgebra spanned by h_i, e_i and f_i, which is isomorphic to $sl_2(F)$. By the finite dimensional representation theory of $sl_2(F)$, h_i is diagonalizable on any finite dimensional $sl_2(F)$-modules. So it is enough to show that the action of g_i on V is locally finite. For any $v \in V$, there exists a positive integer n such that $d\pi(e_i)^n v = 0$. We use induction on n to assert that there exists a non-zero polynomial $f(x) \in F[x]$ such that $f(d\pi(h_i))v = 0$, which is equivalent to the fact that the subspace spanned by $d\pi(h_i)^k v$, $k = 0, 1, 2, \cdots$, is finite dimensional.

Assume $n = 1$. By the above condition ii), there exists a positive integer m such that $d\pi(f_i)^m v = 0$. The fundamental commutator relation in the enveloping algebra $U(sl_2(F))$

$$\left[\frac{e_i^m}{m!}\frac{f_i^m}{m!}\right] = \sum_{j=1}^{m} \frac{f_i^{m-j}}{(m-j)!}\binom{h_i - 2m + 2j}{j}\frac{e_i^{m-j}}{(m-j)!}$$

show that

$$[d\pi(e_i)^m\ d\pi(f_i)^m] = \begin{pmatrix} d\pi(h_i) \\ m \end{pmatrix} v = 0,$$

where for any positive integer j and any integer k,

$$\begin{pmatrix} h_i - k \\ j \end{pmatrix} = \frac{(h_i - k)(h_i - k - 1)\cdots(h_i - k - (j-1))}{j!}.$$

So our assertion follows when $n=1$. For any $v\in V$, by the induction hypothesis, the subspace W spanned by $d\pi(h_i)^k d\pi(e_i)v$, $k=0,1,2,\cdots$, is finite dimensional. Since it is easy to observe that for any k

$$d\pi(e_i)d\pi(h_i)^k v = d\pi(h_i)^k d\pi(e_i)v + d\pi[e_i\, h_i^k]v \in W,$$

there exists a non-zero polynomial $f(x)\in F[x]$ such that $d\pi(e_i)f(d\pi(h_i))v=0$. Our conclusion follows by applying the result obtained in the case of $n=1$ to $f(d\pi(h_i))v$. Then we deduce that the g - invariant subspace

$$U(g)v = \sum_{k,m,n\geq 0} d\pi(f_i)^k d\pi(e_i)^m d\pi(h_i)^n v$$

is finite dimensional for any $v\in V$, since $d\pi(f_i)$ and $d\pi(e_i)$ are locally nilpotent on V, and $d\pi(h_i)$ is locally finite.

Let $\overline{g}'$ denote the Lie algebra on generators e_i, f_i and h_i, $1\leq i\leq n$, with defining relations 1.1) and 1.2). We need the following proposition in order to differentiate modules over the Kac-Moody group G.

Proposition 2.3 Let $(V, d\pi)$ be a $\overline{g}g'$ - module such that all $d\pi(e_i)$ and $d\pi(f_i)$ are locally nilpotent. Then $d\pi(e_i)$ and $d\pi(f_i)$, $1\leq i\leq n$, satisfy relations 1.3) so that we may regard $(v, d\pi)$ as an integrable g' - module.

Proof We will prove that the second relation in 1.3)

$$(\mathrm{ad}\, d\pi(f_i))^{1-a_{ij}} d\pi(f_j) = 0$$

is satisfied if $i\neq j$. The first one in 1.3) follows by the same way. For any $v\in V$, by the relations in 1.1) and 1.2), we have

$$[d\pi(e_i)\ d\pi(f_j)]v = 0\quad [d\pi(h_i)\ d\pi(f_j)]v = -a_{ij}d\pi(f_j)v.$$

Since $d\pi(f_i)$ is locally nilpotent on V, there exists the least positive integer m such that

$$(\mathrm{ad}\, d\pi(f_i))^m d\pi(f_j)v = \sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j \end{pmatrix} d\pi(f_i)^{m-j}d\pi(f_j)d\pi(f_i)^j v = 0.$$

The fundamental commutator relation in $U(sl_2(F))$

$$[e_i\, f_i^m] = -m(m-1)f_i^{m-1} + mf_i^{m-1}h_i$$

implies that

$$ad\,d\pi(e_i)\,(ad\,d\pi(f_i))^m d\pi(f_j)v = m(1-m-a_{ij})\,d\pi(f_i)^{m-1}d\pi(f_j)v = 0,$$

which means $m = 1 - a_{ij}$.

3 Differentiable modules over the Kac-Moody group G

To emphasize the fields, over which the Kac-Moody group and the Kac-Moody algebra are defined, we use $G(F)$ to denote the Kac-Moody group over a field F associated to the Kac-Moody algebra $g'(F)$ over F. Let K be an extension field of F. If $(V, d\pi)$ is an integrable representation of $g'(K)$, it is well known, by the construction of $G(F)$, that there exists the representation π: $G(F)\to GL(V)$ of $G(F)$ given by

$$\pi(U_{a_i}(t)) = \exp td\pi(e_i), \quad \pi(U_{-a_i}(t)) = \circ\exp td\pi(f_i)$$

for any $t\in F$ and $1\le i\le n$. Then the module (V, π) becomes a differentiable module over $G(F)$. In the following, we want to find conditions to determine which modules over $G(F)$ are differentiable.

Lemma 3.1 Let V be a vector space over K, generally speaking, infinite dimensional, and let π: $G(F)\to GL(V)$ be a non-trivial homomorphism. If there exist locally nilpotent linear transformations $d\pi(e_i)$ and $d\pi(f_i)$ on V such that

$$\pi(U_{a_i}(t)) = \exp d\pi(te_i), \quad \pi(U_{a_i}(t)) = \exp d\pi(tf_i)$$

for any $t\in F$ and $1\le i\le n$, $d\pi$ can be extended to an integrable representation of $g'(K)$. Then π can be extended to the homomorphism, denoted again by π, of $G(K)$ into GL(V) so that (V, π) is a differentiable module over $G(K)$.

Proof Set $d\pi(h_i) = [d\pi(e_i)\ d\pi(f_i)]$. What we need to do is to verify that $d\pi(h_i)$, $d\pi(e_i)$ and $d\pi(f_i)$ satisfy relations 1.1), 1.2) and 1.3) for $1\le i\le n$. Let

$$n_i(t) = U_{a_i}(t)U_{-a_i}(-t^{-1})U_{a_i}(t)$$

for any $t\in F^*$. By relation R3),

$$n_i(t)U_{a_i}(u)n_i(t)^{-1} = U_{-a_i}(-t^{-2}u)$$

for $t\in F^*$ and $u\in F$, and

$$U_{-a_i}(-t^{-1})U_{a_i}(u)U_{-a_i}(t^{-1}) = U_{a_i}(-t)U_{-a_i}(-t^{-2}u)U_{a_i}(t).$$

Note that for any locally nilpotent linear transformations $d\pi(x)$ and $d\pi(y)$,

$$(\exp d\pi(x))\,d\pi(y)\,(\exp d\pi(x))^{-1} = \exp ad\,d\pi(x)\,(d\pi(y)),$$

so we have

$$\exp(u\exp ad(-t^{-1}d\pi(f_i))\,d\pi(e_i)) = \exp(u\exp ad(-td\pi(e_i))(-t^{-2}d\pi(f_i))),$$

which yields

$$\exp ad(-t^{-1} d\pi(f_i))\, d\pi(e_i) = -t^{-2} \exp ad(-t d\pi(e_i))\, d\pi(f_i).$$

Expanding two sides and comparing the coefficients of t^{-2} and t^0, we obtain

$$[d\pi(e_i)\ [d\pi(e_i)\ d\pi(f_i)]] = -2d\pi(e_i),$$

$$[d\pi(f_i)\ [d\pi(f_i)\ d\pi(e_i)]] = -2d\pi(f_i),$$

which implies that

$$[d\pi(h_i)\ d\pi(e_i)] = 2d\pi(e_i),\ [d\pi(h_i)\ d\pi(f_i)] = -2d\pi(f_i).$$

Recalling that the subalgebra $g_i(K)$ spanned by e_i, f_i and h_i over K is isomorphic to $sl_2(K)$, so $d\pi$ is a representation of $g_i(K)$, and by Proposition 2.2), $d\pi$ is an integrable representation of $g_i(K)$ on V. Let $V_m = \{v \in V;\ d\pi(h_i)v = mv\}$. Then V admits a direct sum decomposion $V = \oplus_{m\in Z} V_m$ by eigenspaces V_m with respect to $d\pi(h_i)$, and it is clear that $v \in V$ if and only if $\pi(H_{a_i}(t))v = t^m v$ for any $t \in F^*$. Applying R5) to $v \in V_m$, we have

$$t^{-m}\pi(H_{a_i}(t))(v + ud\pi(e_j)v + \cdots) = v + t^{a_{ij}} ud\pi(e_j)v + \cdots,$$

which yields $\pi(H_{a_i}(t))d\pi(e_j)v = t^{a_{ij}+m} d\pi(e_j)v$, for any $t \in F^*$. Then

$$d\pi(h_i)d\pi(e_j)v = (a_{ij} + m)d\pi(e_j)v,$$

which implies that $[d\pi(h_i)\ d\pi(e_j)] = a_{ij}d\pi(e_j) = a_j(h_i)d\pi(e_j)$.

By the same argument, we may deduce

$$[d\pi(h_i)\ d\pi(f_j)] = -a_{ij}d\pi(f_j) = -a_j(h_i)d\pi(f_j).$$

Since the subgroup $H(F)$, generated by $H_{a_i}(t)$ with $t \in F^*$ and $1 \le i \le n$, is abelian, and each $H_{a_i}(t)$ is diagonalizable on V. the subgroup $H(F)$ is diagonalizable on V. Then $d\pi(h_i)$, $1 \le i \le n$, are diagonalizable simultaneously, which means

$$[d\pi(h_i)]\ d\pi(h_j)] = 0.$$

At last, using the same way as it is used at the begining, we know

$$U_{a_i}(t)U_{-a_j}(u)U_{a_i}(-t) = U_{-a_j}(u)$$

implies that

$$[d\pi(e_i)\ d\pi(f_j)] = 0$$

for $i \neq j$. Hence $d\pi$ can be extended to a representation of $\overline{g}'(K)$, and by Proposition 2.3), $d\pi$ is an integrable representation of $g'(K)$. □

We now can give a characterizatin of differentiable modules over the Kac-Moody group $G(F)$.

Theorem 3.2 Let V be a vector space over F. (V, π) is a differentiable module over the

Kac- Moody group $G(F)$ if and only if the restriction of π to each U_a with $a=\pm a_i$ is a rational representation of U_a.

Proof The root subgroup U_a, which is isomorphic to the additive group of the field F, is a unipotent algebraic group over F. Hence, for any finite dimensional polynomial representation ρ,

$$\beta(U_a(t)) = \exp(tA)$$

for some unique nilpotent linear trasformation A. (see[6], Ch. 8, Th. 1. 2) Then there exist locally nilpotent linear transformations $d\pi(e_i)$ and $d\pi(f_i)$ on V such that

$$\pi(U_{a_i}(t)) = \exp\ td\pi(e_i), \quad \pi(U_{-a_i}(t)) = \exp\ td\pi(f_i)$$

for any $t\in F$. By Lemma 3. 1, the theorem follows □

It is well known that the Kac- Moody group $G(F)$ associated to a Cartan matrix is the universal Chevalley group. For the Chevalley group $G(Q)$ over the rational field Q, an interesting result is that all finite dimensional modules over $G(Q)$ are differentiable. For $G=SL_n(Q)$, the result was established by Steinberg[7]. In the following, we will establish the result for any Chevalley group $G(Q)$. Let $G(F)$ denote the universal Chevalley group over a field F of characteristic zero. Following the standard notation, we write $U(F)$ for the maximal unipotent subgroup of $G(F)$ generated by root subgroups corresponding to positive roots, and $B(F)$ for the Borel subgroup of $G(F)$ generated by $H(F)$ and $U(F)$.

Lemma 3. 3 Let F and K be any fields of characteristic zero. For any homomorphism

$$\pi: G(F)\rightarrow SL_n(K),$$

$\pi(U(F))$ is a unipotent subgroup in $SL_n(K)$.

Proof A subgroup A in $SL_n(K)$ is unipotent if and only if there exists a positive integer r such that whenever $a_1, a_2, \ldots, a_r\in A$, then

$$(a_1 - 1)(a_2 - 1)\cdots(a_r - 1) = 0.$$

So, without losss of generality, we may assume that K is algebraically closed. Write $\pi(B(F))=A$. Since $B(F)$ is solvable, so is A. Let $\bar{A}^0$ denote the connected component of the Zariski closure of A in $SL_n(K)$. Then $\bar{A}^0$ is a connected solvable algebraic subgroup of $SL_n(K)$, and its commutator subgroup is unipotent. Let $S=\pi^{-1}(A\cap\bar{A}^0)$. Then S is a subgroup of $B(F)$ of finite index. Since each root subgroup $U_a(F)$ is isomorphic to the additive group of F, $U_a(F)$ has no proper subgroup of finite index, hence $U(F)\subseteq S$. Then it will suffice to show that $U(F)\subseteq S'$, but by relation R5) this fact is clear since $H(F)\cap S$ is a subgroup of $H(F)$ of finite index. □

Theorem 3. 4 Let V be a finite dimensional vector space over F and

$$\pi: G(Q)\rightarrow SL(V)$$

a nontrivial homomorphism. Then π can be extended to a homomorphism, denoted again by π, of $G(F)$ into $SL(V)$; moreover the module (V,π) over $G(F)$ is differentiable.