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微分包含的稳定性

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摘 要

本文通过引用生存定理, Lyapunov 第二方法和比较定理, 首先证明了微分包含的整体存在定理, 然后讨论了微分包含解的稳定性

Journal of Mathematical Research & Exposition
Vol. 17, No. 1, 43--49, Feb. 1997

Stability for Differential Inclusions*

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Abstract In this paper, we first prove some global existence theorems for differential inclusions by using viability theorem, Lyapunov's second method and comparison theorem, and then discuss the stability of solutions for differential inclusion.

Keywords differential inclusion, stability, global existence, viability theorem.
Classification AMS(1991) 49J24/CCL O176

1. Introduction

In recent years the study of differential inclusions has been devoloped considerably, with applications to mathematical economics[1],[2], nonsmooth dynamics and optimal control[4], etc. Also differential inclusion provides a mathematical tool for studing differential equation with a discontinuous right handside[6].

Recently, Seach[10] and Taniguchi[12] proved that the differential inclusion

$$\overset{\circ}{x} \in F(t, x),\ x(0) = x_0 \qquad (*)$$

has global solution defined on $[t_0, \infty)$ and they also considered asymptotic equilibrium of solution for differential inclusion (*). Roxi[9] considered the weakly (strongly) stability for general control systems. Along the same lines, T. F. Bridgland. Jr[3] studied weak stability of solution for differential inclusion (*) under some strongly assumptions. We also mention that [8] discussed stability for differential equatoin with discontinuous right-hand sides.

The purpose of this paper is to present global existence theorems and stability of solution for differential (*) by using viability theorem, Lyapunov's second method and comparison theorem.

2. Global existence of solution to differential inclusion

Let $K \subset R^n$ be a subset of R^n and $x \in K$. The contingent cone $T_K(x)$ is defined by

$$T_K(x) = \{v \mid \liminf_{h \to 0^+} d_K(x + hv)/h = 0\}.$$

* **Received Mar.** 14, 1994

A set-valued map $F: R^n \to R^m$ is called upper semicontinuous at $x \in \mathrm{Dom}(F)$ if and only if for any neighborhood U of $F(x)$ there exists $\eta > 0$ such that for every $x' \in B(x, \eta)$, $F(x') \subset U$. It is said to be upper semicontinuous if and only if it is upper semicontinuous at any point of Dom (F).

Let (Ω, Σ) be a measurable space and $F: \Omega \to R^n$ be a set-valued map with closed images. The map F is called measurable if the inverse image of each open set is measurable set.

A set-valued map $F: R \times R^n \to R^n$ is said to be linear growth if there is $C(t) \in L^1(R^+, R^n)$ such that

$$\|F(t, x)\| = \sup \|v\| \, |v \in F(t, x) \le C(t)(1 + \|x\|) \text{ on } R^+ \times R^n.$$

Let $V: R^n \to R \cup \{+\infty\}$ be a nontrivial extended function and x belong to its domain. We shall say that the function $D_\uparrow v(x)$ from R^n to $R \cup +\infty$ defined by

$$D_\uparrow V(x)(u) = \lim_{h\to 0} \inf_{u'\to u} (1/h)(V(x + hu') - V(x))$$

is the contingent epiderivative of V at x in the direction u.

The function V is contingently epidifferentiable at x if and only if $D_\uparrow V(x)(0) = 0$. It is said episleek (at x) if its epigraph is sleek (at $(x, V(x))$) i.e., $T_{\mathrm{ep}V}(x, V(x))$ is upper semicontinuous at $(x, V(x))$.

If V is Lipsctiz at a point of its domain, then $D_\uparrow V(x)(u)$ is the lower Dini-derivative.

We define in a similar way the contingent hypoderivative $D_\downarrow V(x)$ from $R^n \to R \cup +\infty$ of V: $R^n \to R \cup +\infty$ at a point x of its domain by

$$D_\downarrow V(x)(u) = -D_\uparrow(-V)(x)(u) = \lim_{h\to 0} \sup_{u'\to u} (1/h)(V(x + hu') - V(x)).$$

We consider the differential inclusion

$$\mathring{x} \in F(t, x(t)), \quad x(0) = x_0, \tag{2.1}$$

and the comparison differential equation

$$\mathring{u}(t) = g(t, u), \quad u(t_0) = u_0, \tag{2.2}$$

where $F: R \times R^n \to R^n$ is a set-valued map, which satisfies that

(a) for each $x \in R^n$, $t \to F(t, x)$ is measurable,

(b) for almost $t \in R^+$, $x \to F(t, x)$ is upper semicontinuous;

and $g: R \times R^1 \to R^+$ satisfies the Caratheodory conditions.

A function $x(\bullet)$ is called a solution of differential inclusion (2.1) or differential equation (2.2) if $x(\bullet)$ is absolutely continuous and satisfies (2.1) or (2.2) for almost all t respectively.

Theorem 1 *Let $V: R^+ \times R^n \to R \cup +\infty$ be a nonnegative contingently epi-differentable lower semicontinuous extended function and $F: R^+ \times R^n \to R^n$ be a nontrivial upper semicontinuous set-valued map with compact convex images. We suppose that*

(i) for every $(t, x) \in \mathrm{dom}(V)$, $\inf_{v \in F(t,x)} D_\uparrow V(t, x)(1, v) \le g(t, V(t, x))$,
where $g \in C(R^+ \times R^1 \to R)$;

(ii) *for any given* $[0, T]\subset R^+$, $\lim_{|x|\to+\infty} V(t, x) = +\infty$ *uniformly for* $t\in[0, T]$.

If the differential equation (2.2) *has a maximal solution* $r(t, t_0, u_0)$ *defined on* $[t_0,\infty)$, *then for any* x_0 *such that* $V(t_0, x_0)\le u_0$, *there exists a solution* $x(t)$ *of the differential inclusion* (2.1) *defined on* $[t_0,\infty)$ *such that* $V(t, x(t))\le r(t)$, $t\ge t_0$.

Proof We set $G(t, x, u) = F(t, x)\times g(t, u)$, $K(t) = \mathcal{E}p V(t)$. Since $F(t, x)$ is compact and $DV(t, x)(v)$ is lower semicontinuous, (i) implies that there is $v\in F(t, x)$ such that (see Prop. 6.14 of [2])

$$(1, v, g(t, V(t, x))\in T_{\mathcal{E}p(V)}(t, x, V(t, x)) = \mathcal{E}p(DV(t, x)).$$

Thus $z_n := (t+h_n t_n, x_n+h_n v_n, V(t, x)+h_n s_n)\in \mathcal{E}p(V)$ with $h_n\to 0_+$, $t_n\to t$, $v_n\to v$ *and* $s_n\to g(t, V(t, x))$. *If* $u>V(t, x)$, this implies that for large n.

$$(t+h_n t_n, x_n+h_n v_n, u+h_n g(t, u))$$
$$= z_n + (0, 0, u-V(t, x)-h_n(s_n-g(t, u)))\in \mathcal{E}p(V) + \{0\}\times\{0\}\times R^+ = \mathcal{E}p(V), \text{ so that}$$
$$(1, v, g(t, u))\in T_{\mathcal{E}p(V)}(t, x, u)),$$

i.e., $(1, G(t, x, u)\cap T_{\mathrm{Graph}(K)}(t, x, u)\ne\emptyset$.

By Theorem 3.25 of [1], for initial state $(t_0, x_0, u_0)\in\mathrm{Graph}(K)$, there exists a positive T and a variable solution (x, u) on $[t_0, T]$ to differential inclusion

$$(\dot{x}(t), \dot{u}(t))\in G(t, x(t), u(t))$$

such that

$$(x(t), u(t))\in K(t),\ t\in[t_0, T],\ \text{i.e.},\ V(t, x(t))\le u(t),\ t\in[t_0, T],$$

and

$$\text{either } T = +\infty \text{ or } T<\infty \text{ and } \limsup_{t\to T^-}(\|x(t)\| + \|u(t)\|) = +\infty.$$

Since the maximal solution $r(t, t_0, u_0)$ for differential equation (2.2) is defined on $[t_0,\infty]$ and so $V(t, x)\le u(t)\le r(t, t_0, u_0)$. This and (ii) imply $T = +\infty$. Otherwise, the boundedness of $r(t)$ on $[t_0, T)$ and (ii) imply that $\|x(t)\|$ is bounded on $[t_0, T)$. This is a contradiction. □

Remark A similar result was discussed in [1] where F was assumed to be linear growth and was indepent of t.

As a consequence, we have

Corollary 1 *Let* F *be as in Theorem 1 and* $g: R^+\times R^+\to R^+$ *be a continuous functions. Assume that*

(iii) $d(0, F(t, x))\le g(t, \|x\|)$, $t\ge t_0$.

If the differential equation (2.2) *has a maximal solution* $r(t, t_0, u_0)$ *defined on* $[t_0,\infty)$, *then for any* x_0, $|x_0|\le u_0$, *there exists a solution* $x(t)$ *of the differential inclusion* (2.1) *defined on* $[t_0,\infty)$ *such that* $\|x(t)\|\le r(t)$, $t\ge t_0$.

Proof Let $V(t, x) = \|x\|$. We have

$$\inf_{v\in F(t,x)} D_\uparrow V(t,x)(1,v) = \inf_{v\in F(t,x)} \sup_{x^*\in\partial\|x\|} \langle x^*, v\rangle \le \inf_{v\in F(t,x)} \|v\| \le g(t, \|x\|).$$

Since (see [4])

$$\partial\|x\| = \begin{cases} \{x^i/\|x\|\}, & \text{if } x \ne 0, \\ x^* \in R^n \mid \|x^*\| \le 1, & \text{if } x = 0. \end{cases}$$

This completes the proof of the Corollary by using Theorem 1. □

Theorem 2 *Let $V: R^+ \times R^n \to R \cup +\infty$ be a nonnegative contingently epi-differentiable episleek lower semicontinuous extended function. Let $F: R^+ \times R^n \to R^n$ be linear growth set-valued map with compact convex images and $g: R^+ \times R^+$ be linear growth nonnegative function. We suppose that F is measurable in t, upper semicontinuous in x and g satisfies Caratheodory conditions. Assume also that (i) and (ii) hold. If the differential equation (2.2) has a maximal solution $r(t, t_0, u_0)$ defined on $[t_0, \infty)$, then for any x_0, $V(t_0, x_0) \le u_0$, there exists a solution $x(t)$ of the differential inclusion (2.1) defined on $[t_0, \infty)$ such that $V(t, x(t)) \le r(t)$, $t \ge t_0$.*

Proof We can proves the Theorem by using Theorem 1 of [5] and the similar method in proof of Theorem 1.

Remark Theorem 2 remains true for non-episleek $V(t, x)$, if we replace epiderivative $D_\uparrow V(t, x)(1, v)$ in (i) by Circatangent epiderivative $D_\uparrow V(t, x)(1, v)$ (see [2] and [4]).

Theorem 3 *Let F, g be as in Theorem 1 and let $V: R^+ \times R^n \to R \cup +\infty$ be a nonnegative contingently hypodifferentable continuous function. Assume that F is linear growth and that (ii) and*

(*iv*) *for every* $(t, x) \in \mathrm{dom}(V)$, $\sup_{v\in F(t,x)} D_\uparrow V(t, x)(1, v) \le g(t, V(t, x))$.

If the differential equation (2.2) has a maximal solution $r(t, t_0, u_0)$ defined on $[t_0, \infty)$, then for any $x_0, V(t_0, x_0) \le u_0$, all solutions $x(t)$ of the differential inclusion (2.1) defined on $[t_0, \infty)$ such that $V(t, x(t)) \le r(t)$, $t \ge t_0$.

Proof By Theorem 1 the differential inclusion (2.1) has a solution defined on $[t_0, \infty)$ such that $V(t, x(t)) \le r(t)$, $t \ge t_0$.

Let $x(\cdot)$ be an arbitrary solution to the differential inclusion (2.1). We assert that its maximal existence interval is $[t_0, \infty)$ and $V(t, x(t)) \le r(t)$, $t \ge t_0$. Indeed, if its maximal existence interval is $[t_0, T)$, $T < +\infty$, *for any* $t \in [t_0, T)$, the upper semicontinuty of F implies

$$x(t+h) - x(t) \in hF(t, x) + \epsilon hB$$

for small h, where B denotes the unit ball of R^n. This and (iv) yield

$$D_\downarrow V(t, x(t)) = \limsup_{h\to 0} [V(t+h, x(t+h)) - V(t, x(t))]/h$$

$$\le \sup_{v\in F(t,x(t))} D_\downarrow V(t, x(t))(1, v) \le g(t, V(t, x)).$$

Thus Theorem 1.2.6 of [7] (see also [11]) implies $V(t, x(t)) \le r(t)$, $t \in [t_0, T)$. Since $r(t)$ is bounded on $[t_0, T)$ and $\lim_{\|x\|\to\infty} V(t, x) = +\infty$ uniformaly on $[t_0, T)$, then $V(t, x(t)) \le r(t)$, $t \in [t_0, T)$ implies that $\|x(t)\|$ is bounded on $[t_0, T)$ and so there exists $M > 0$ such that $\|x(t)\| \le M$ for $t \in [t_0, T)$. Hence $\|F(t, x(t)\| - \sup_{v \in F(t, x(t))} \|v\| \le (M+1)C(t)$, and then for any $t_0 \le t_1 \le t_2 \le T$ we have

$$\|x(t_2) - x(t_1)\| \le (M+1)\int_{t_1}^{t_2} C(t)\,dt$$

and by the absolutely continuous of integral, $x(t)$ has a limit when $t \to T^-$, we denote this limit by z_0.

By Theorem 3.2.5 of [1] there exists $T_1 > T$ and a solution $x_1(t)$ starting at z_0 to differential inclusion $x(t) \in F(t, x(t))$ on $[T, T_1)$.

Define

$$z(t) = \begin{cases} x(t), & \text{if } t \in [t_0, T), \\ x_1(t), & \text{if } t \in [T, T_1). \end{cases}$$

Then $z(t)$ is a solution of the differential inclusion (2.1) defined on $[t_0, T_1)$. This is a contradiction. □

3 Stability of differential inclusion

In this section we assume that

(v) $0 \in F(t, 0)$, $g(t, 0) = 0$ and $V(t, 0) = 0$, $t \in R^+$

hold. Consider differential inclusion

$$\mathring{x}(t) \in F(t, x), \tag{3.1}$$

and differential equation

$$\mathring{u}(t) = g(t, u). \tag{3.2}$$

The trivial solution $x = 0$ of (3.1) is stable (weakly stable), if for each $\epsilon > 0$, $t_0 \in R^+$, there exists $\delta = \delta(t_0, \epsilon) > 0$ such that for each x_0, $\|x_0\| < \delta$, all (there exists a) solution $x(t, t_0, x_0)$ of (3.1) satisfying $\|x(t, t_0, x_0)\| < \epsilon$, $t \ge t_0$.

If δ is independent of t_0, we say that the solution $x = 0$ of (3.1) is uniformly stable (uniformly weakly stable). Other stability notions (equistable, asymptotic stable, equiasymptotic stable) can be similarly defined (see [7], [11]).

The trivial solution $u = 0$ of (3.2) is stable, if for each $\epsilon > 0$, $t_0 \in R^+$, there exists $\delta = \delta(t_0, \epsilon) > 0$ such that for each u_0 $\|u_0\| < \delta$, there exists a maximal solution $r(t, t_0, u_0)$ of (3.2) satisfying $|r(t, t_0, u_0)| < \epsilon$, $t \ge t_0$.

Other stability of $u = 0$ to the differential equation (3.2) can be similarly defined (see [7] and [11]).

Theorem 4 *Let F and g be as in Theorem 1 (or Theorem 2) and let $V: R^+ \times R^n \to R^+$ be a contingently epidifferentable continuous function. Assume that (i), (ii) hold and*

(vi) there exists a continuous strict increasing function Φ on R^+ satisfying $\Phi(0) = 0$ and $\Phi(\|x\|) \le V(t, x)$, $(t, x) \in R^+ \times R^n$.

Then the following conclusion holds

1.° *If the trivial solution* $u=0$ *of* (3.2) *is stable* (*or asymptotic stable*), *then the trivial solution* $x=0$ *of* (3.1) *is weakly stable* (*or asymptotic weakly stable*).

Furthermore, assume that

(*vii*) *there exists a continuous strict increasing function* Ψ *on* R^+ *satisfying* $\Psi(0)=0$ *and* $V(t,x)\le\Psi(\|x\|)$, $(t,x)\in R^+\times R^n$.

2.° *If the trivial solution* $u=0$ *of* (3.2) *is uniformly stable* (*or uniformly asymptotic stable*), *then the trivial solution* $x=0$ *of* (3.1) *is uniformly weakly stable* (*or uniformly asymptotic weakly stable*) *respectively.*

Proof For any given $(t_0,x_0)\in R^+\times R^n$, by Theorem 1 (or Theorem 3), there exists a solution $x(t,t_0,x_0)$ of the differential inclusiton (3.1) and a solution $u(t,t_0,V(t_0,x_0))$ of the differential equation (3.2) such that

$$V(t,x(t,t_0,x_0))\le u(t,t_0,V(t_0,x_0))\le r(t,t_0,V(t_0,x_0)),\qquad(3.3)$$

and (vi) implies

$$\Phi(\|x(t,t_0,x_0)\|)\le V(t,x(t,t_0,x_0)).\qquad(3.4)$$

If the trivial solution $u=0$ is stable, then for any $\epsilon>0$, $t_0\in R^+$, there exists $\delta^*=\delta^*(t_0,\epsilon)>0$ such that for each u_0, $0\le u_0<\delta^*$, we have

$$0\le r(t,t_0,u_0)<\Phi(\epsilon),\quad t\ge t_0.\qquad(3.5)$$

From the continuity of V, there exists $\delta(t_0,\epsilon)>0$ such that for each $\|x_0\|<\delta(t_0,\epsilon)$ we have $0\le V(t_0,x_0)<\delta^*(t_0,\epsilon)$ and so $0\le r(t,t_0,V(t_0,x_0))<\Phi(\epsilon)$. This and (3.3), (3.4) imply that for $\|x_0\|<\delta(t_0,\epsilon)$ we have $\|x(t,t_0,x_0)\|<\epsilon$. Therefore the solution x=0 of differential inclusion (3.1) is weakly stable.

It is easy to show that the other conclusions hold by the standard argument.

If we replace the assumptions on F and g of Theorem 4 by assuming that F and g satisfy the conditions of Theorem 3, we can prove the corresponding results for strongly stability properties

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