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A Class of Rational Arithmetical Functions with Combinatorial Meanings*

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Abstract An arithmetical function f is said to be a rational arithmetical function of order (s, r) if there exist completely multiplicative functions $f_1, f_2, \cdots, f_s$ and $g_1, g_2, \cdots, g_r$ such that

$$f = f_1 * f_2 * \cdots * f_s * (g_1)^{-1} * (g_2)^{-1} * \cdots * (g_r)^{-1},$$

where $*$ is the Dirichlet convolution. Recently, L. C. Hsu and Wang Jun studied combinatorial meanings of rational arithmetical functions of order $(1, r)$. We study these meanings in the setting of Narkiewicz's regular convolution.

Keywords Euler totient function, regular convolutions, multiplicative functions

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1. Introduction

An arithmetical function f is said to be multiplicative if f is not identically zero and

$$f(mn) = f(m)f(n) \tag{1.1}$$

whenever $(m, n) = 1$. A multiplicative function f is said to be completely multiplicative if (1.1) holds for all m and n. The Dirichlet convolution of two arithmetical functions f and g is defined by

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d).$$

A multiplicative function f is said to be a rational arithmetical function of order (s, r) if there exist completely multiplicative functions $f_1, f_2, \cdots, f_s$ and $g_1, g_2, \cdots, g_r$ such that

$$f = f_1 * f_2 * \cdots * f_s * (g_1)^{-1} * (g_2)^{-1} * \cdots * (g_r)^{-1}.$$

This concept originates with Vaidyanathaswamy[10]. General properties of these functions can be found, e.g., in [1], [4], [8] and [10].

In particular, rational arithmetical functions of order (2, 0) are said to be specially multi-

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plicative functions or quadratic functions, and rational arithmetical functions of order (1, 1) are said to be totient functions, see e. g., the books by P. J. McCarthy[6] and R. Sivaramakrishnan[9].

It is well known that various totient functions are combinatorial number-theoretic functions in character. Recently, L. C. Hsu and Wang Jun[5, 11] studied combinatorial meanings of rational arithmetical functions of order $(1, r)$. These functions may also be referred to as totient functions of order r.

In [3], the author generalizes the concept of a rational arithmetical function of order (s, r) to the setting of Narkiewicz's regular convolution. These functions are referred to as A-rational arithmetical functions of order (s, r). For definition, see Section 2. In [3], the author studies, among other things, basic properties of A-rational arithmetical functions of order (2, 0) and (1, 1), see also [6, Chapter 4].

The purpose of this paper is to study combinatorial meanings of A-rational arithmetical functions of order $(1, r)$. The idea for these combinatorial meanings arises from the papers [5, 11]. In fact, we present certain results of [5, 11] in the setting of Narkiewicz's regular convolution, see Remarks 4. 2 and 4. 3 of this paper.

2 Regular Convolutions and the Related Rational Arithmetical Functions

In this section we introduce the concept of Narkiewicz's regular convolution. Back-ground material on regular convolutions can be found e. g. in [6, Chapter 4] and [7]. We here review the concepts and notations, which are needed in this paper.

For each n, let $A(n)$ be a subset of the set of positive divisors of n. The elements of $A(n)$ are said to be the A-divisors of n. The A-convolution of two arithmetical functions f and g is defined by

$$(f *_A g)(n) = \sum_{d \in A(n)} f(d) g(n/d).$$

Narkiewicz[7] defines an A-convolution to be regular if

(a) the set of arithmetical functions forms a commutative ring with unity with respect to the ordinary addition and the A-convolution,

(b) the A-convolution of multiplicative functions is multiplicative,

(c) the function $\equiv 1$ has an inverse μ_A with respect to the A-convolution, and $\mu_A(n) = 0$ or -1 whenever n is a prime power.

The inverse of an arithmetical function f such that $f(1) \neq 0$ with respect to the A-convolution is defined by

$$f *_A f^{-1} = f^{-1} *_A f = \delta,$$

where $\delta(1) = 1$ and $\delta(n) = 0$ for $n > 1$. It can be proved [7] that an A-convolution is regular if and only if

(i) $A(mn) = \{de: d \in A(m), e \in A(n)\}$ whenever $(m, n) = 1$,

(ii) for each prime power $p^a (> 1)$ there exists a divisor $t = \tau_A(p^a)$ of a such that

$$A(p^a) = \{1, p^t, p^{2t}, \cdots, p^{rt}\},$$

where $rt = a$, and

$$A(p^{it}) = \{1, p^t, p^{2t}, \cdots, p^{it}\}, 0 \le i < r.$$

The positive integer $t = \tau_A(p^a)$ in item (ii) is said to be the A-type of p^a. A positive integer n is said to be A-primitive if $A(n) = \{1, n\}$. The A-primitive numbers are 1 and p^t, where p runs through the primes and t runs through the A-types of the prime powers p^a with $a \ge 1$. The order of an A-primitive number $p^t (> 1)$ is defined by

$$o(p^t) = \sup\{s \in \mathbf{Z}^+ : \tau_A(p^{st}) = t\}.$$

For all n, let $D(n)$ be the set of all positive divisors of n and let $U(n)$ be the set of all unitary divisors of n, that is,

$$U(n) = \{d > 0 : d \mid n, (d, n/d) = 1\} = \{d > 0 : d \parallel n\}.$$

The D-convolution is the classical Dirichlet convolution and the U-convolution is the unitary convolution [2]. These convolutions are regular with $\tau_D(p^a) = 1$ and $\tau_U(p^a) = a$ for all prime powers $p^a (> 1)$. Further, if $A = D$, then $o(p) = \infty$ for all primes p, and if $A = U$, then $o(p^a) = 1$ for all prime powers $p^a (> 1)$.

We say that an integer $n (> 1)$ is (A, r)-powerful if for each A-primitive prime power $p^t \in A(n)$ we have $o(p^t) \ge r$ and $p^{rt} \in A(n)$. If $A = D$, then (A, r)-powerful numbers are the usual r-powerful numbers. If $A = U$ and $r > 1$, then no integer $n (> 1)$ is (A, r)-powerful. If $r = 1$, then all integers $n (> 1)$ are (A, r)-powerful for each regular convolution A.

The A-analogue of the Möbius function μ_A is the multiplicative function given by

$$\mu_A(p^a) = \begin{cases} -1 & \text{if } p^a (> 1) \text{ is } A\text{-primitive}, \\ 0 & \text{if } p^a \text{ is non-}A\text{-primitive}. \end{cases}$$

In particular, $\mu_D = \mu$, the classical Möbius function, and $\mu_D = \mu^*$, the unitary analogue of the Möbius function [2].

Let A be a regular convolution. An arithmetical function f is said to be A-multiplicative [12] if f is not identically zero and

$$f(n) = f(d) f(n/d)$$

whenever $d \in A(n)$.

Definition *An arithmetical function f is said to be an A-rational arithmetical function of order (s, r) if there exist A-multiplicative functions $f_1, f_2, \cdots, f_s$ and $g_1, g_2, \cdots, g_r$ such that*

$$\begin{aligned} f &= f_1 *_A f_2 *_A \cdots *_A f_s *_A (g_1)^{-1} *_A (g_2)^{-1} *_A \cdots *_A (g_r)^{-1} \\ &= f_1 *_A f_2 *_A \cdots *_A f_s *_A (\mu_A g_1) *_A (\mu_A g_2) *_A \cdots *_A (\mu_A g_r) \end{aligned} \tag{2.1}$$

(*see* [3]).

The D-rational arithmetical functions of order (s, r) are the usual rational arithmetical functions of order (s, r).

3 A Combinatorial Meaning

Let u be an arbitrary but fixed positive integer. We denote u-vectors of integers as $\mathbf{a} = \langle a_1, a_2, \cdots, a_u \rangle$, and we write $\mathbf{a} \equiv b \pmod n$ if $a_i = b_i$ for all $i = 1, 2, \cdots, u$.

Let r be a fixed positive integer and let $p^t (> 1)$ be an A-primitive prime power. For each $i = 1, 2, \cdots, r$, let $S_i(p^t)$ be a subset of $\mathbf{Z}_{p^t}^u$, where $\mathbf{Z}_{p^t}^u = \{\mathbf{a}: 0 \le a_i < p^t, i = 1, 2, \cdots, u\}$, and let $f_i(p^t)$ denote the cardinality of $S_i(p^t)$.

Definition *The arithmetical function $\Phi_{S,r}$ is defined by*

$$\begin{aligned}\Phi_{S,r}(n) &= \sum_{d \in A(n)} (n/d)^u [\mu_A f_1) *_A (\mu_A f_2) *_A \cdots *_A (\mu_A f_r)](d) \\ &= [E^u *_A (\mu_A f_1) *_A (\mu_A f_2) *_A \cdots *_A (\mu_A f_r)](n),\end{aligned} \tag{3.1}$$

where $E^u(n) = n^u$ for n and $f_1, f_2, \cdots, f_r$ are A-multiplicative functions defined by

$$f_i(n) = \prod_{p^t \in A(n)} f_i(p^t)^{\vartheta_p(n)/t}, \ i = 1, 2, \cdots, r,$$

$(n = \prod_p p^{\vartheta_p(n)})$.

The function $\Phi_{S,r}(n)$ is an A-rational arithmetical function of order $(1, r)$. It can be verified that if n is (A, r)-powerful, then $\Phi_{S,r}(n)$ can be written as

$$\Phi_{S,r}(n) = n^u \prod_{p^t \in A(n)} \prod_{i=1}^{r} \left(1 - \frac{f_i(p^t)}{p^{tu}}\right) \tag{3.2}$$

We now introduce a combinatorial meaning of the function $\Phi_{S,r}(n)$. Let $p^t (> 1)$ be an A-primitive prime power such that $o(p^t) \ge r$. For every u-vector $\mathbf{a}$ there is a unique $r \times u$ matrix $M_{p^t}(a)$ over $\mathbf{Z}_{p^t}$ such that

$$a \equiv (1, p^t, p^{2t}, \cdots, p^{(r-1)t}) M_{p^t}(a) \pmod{p^{rt}}.$$

Definition *We write $(\mathbf{a}, n)_{S,r} = 1$ if for each $i = 1, 2, \cdots, r$, the ith row of $M_{p^t}(\mathbf{a})$ is not in $S_i(p^t)$ for every $p^t \in A(n)$.*

Theorem 1 *Let n be an (A, r)-powerful number. Then the number of u-vectors $\mathbf{a} \pmod n$ such that $(\mathbf{a}, n)_{S,r} = 1$ is equal to $\Phi_{S,r}(n)$.*

Proof Let $n = p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}$ be an (A, r)-powerful number. Let $N_{S,r}(n)$ denote the number of u-vectors a $\pmod n$ such that $(a, n)_{S,r} = 1$. We show that $N_{S,r}(n)$ is multiplicative. By the Chinese remainder theorem, for each ordered set of u-vectors $a_1, a_2, \cdots, a_m$ there is a unique u-vector a $\pmod n$ such that

$$a \equiv a_1 \pmod{p_1^{e_1}}, a \equiv a_2 \pmod{p_2^{e_2}}, \cdots, a \equiv a_m \pmod{p_m^{e_m}}.$$

Conversely, for any u-vector a $\pmod n$ there are unique $a_i \pmod{p_i^{e_i}}$, $i = 1, 2, \cdots, m$, satisfying the above system of congruences. Therefore we can conclude that $(a, n)_{S,r} = 1$ if and only if $(a_i, p_i^{e_i})_{S,r}$

$= 1$ for all $i = 1, 2, \cdots, m$. This proves that $N_{S,r}(n)$ is multiplicative.

We next consider the value of $N_{S,r}(p^e)$. Denote $t = \tau_A(p^e)$ and $p^e = p^{st}$. Since n is (A, r)-powerful, we have $s \geq r$. Let a(mod p^{st}) be any u-vector such that $(\mathrm{a}, p^{st})_{S,r} = 1$. Then

$$\mathrm{a} = (1, p^t, p^{2t}, \cdots, p^{(r-1)t}) M_{p^t}(a) + p^{rt} a',$$

where a$'$ is a u-vector (mod $p^{(s-r)t}$). Clearly, $(a, p^{st})_{S,r} = 1$ if and only if for every $i = 1, 2, \cdots, r$, the ith row vector of $M_{p^t}(a)$ is in $Z^u_{p^t} \setminus S_i(p^t)$. From this we can see that the number of u-vectors a (mod p^{st}) such that $(\mathrm{a}, p^{st})_{S,r} = 1$ is given by

$$N_{S,r}(p^{st}) = p^{(s-r)tu} \prod_{i=1}^{r} (p^{tu} - f_i(p^t)) = p^{stu} \prod_{i=1}^{r} \left((1 - \frac{f_i(p^t)}{p^{tu}} \right)$$

By multiplicativity, we have

$$N_{S,r} = n^u \prod_{p^t \in A(n)} \prod_{i=1}^{r} \left((1 - \frac{f_i(p^t)}{p^{tu}} \right) = \varphi_{S,r}(n).$$

This completes the proof.

Remark 3.1 The combinatorial meaning of the function $\varphi_{S,r}(n)$ is restricted to (A, r)-powerful numbers. If $r = 1$, then $\varphi_{S,r}(n)$ is an A-totient function and the combinatorial meaning holds for all n.

Example 3.1 Let $u = 1$ and $S_i(p^t) = \{0\}$ for all $i = 1, 2, \cdots, r$. Let n be an (A, r)-powerful number. Then for any integer a, we have $(a, n)_{S,r} = 1$ if and only if for each A-primitive prime power $p^t \in A(n)$,

$$a \equiv a_0 + a_1 p^t + \cdots + a_{r-1} p^{(r-1)t} (\mathrm{mod}\, p^{rt}),$$

where $0 < a_i < p^t$ for all $i = 0, 1, \cdots, r-1$. By (3.1) and (3.2),

$$\varphi_{S,r}(n) = \sum_{d \in A(n)} (n/d) \mu_r(d) = n \prod_{p^t \in A(n)} (1 - \frac{1}{p^t})^r,$$

where

$$\mu_r(d) = [(\mu_A) *_A (\mu_A) *_A \cdots *_A (\mu_A)](d) = \prod_{p^t \in A(d)} \binom{r}{\partial_p(d)/t} (-1)^{\partial_p(d)/t}.$$

The functions $\varphi_{S,r}$ (resp. μ_r) may be referred to as the A-analogue of the Euler totient (resp. M bius function) of order r.

Remark 3.2 Let $A = D$ in Example 3.1. Then the function $\varphi_{S,r}$ becomes the Euler totient φ_{μ_r} of order r, see Wang and Hsu [11, Theorem 3.1]. Further, $(a, n)_{S,r} = 1$ means that a is r-th degree prime to n in the terminology of Wang and Hsu [11, Definition 3.1]. If, in addition, $r = 1$, then the function $\varphi_{S,r}$ reduces to the classical Euler totient φ and $(a, n)_{S,r} = 1$ means that $(a, n) = 1$, that is, a is prime to n.

Remark 3.3 If $A = D$ and $S_1 = S_2 = \cdots = S_r$, then Theorem 1 of this paper becomes Theorem 3.3

of [11]. If $A = D$ and $u = 1$, then Theorem 1 of this paper becomes Theorem 2.1 of [5].

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