This is a contradiction. Hence $V_n(x)$ is a neighbourhood of x. Thus X is a first countable space. Since, a quasi-base is a K-network, from Theorem A (iii), we know that X is a Lašnev space, so X is metrizable.

Corrllary 2.4 *If X has a σ-CF quasi-base, then the following statements are equivalent*:

(*i*) *X is metrizable.*

(*ii*) *X is a Lašnev space.*

(*iii*) *X is a Fréchet space.*

(*iv*) *X is a k'-space.*

Remark It is known that if a k-space has a σ-CF base, then it is metrizable. So, a question may be raised: If a k-space has a σ-CF quasi-base, is it metrizable? The question is interesting since we have already known the relation: $k'+$ (σ-CF quasi-base) $\Leftrightarrow k+$ (σ-CF base). However, is this balance relation essential?

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关于Lašnev 空间的超空间

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摘 要

讨论了Lašnev 空间的超空间的某些性质. 文中构造一反例, 证明存在可数Lašnev 空间, 其紧子集超空间不是层型空间. 并指出文[6]中关于上述结果的证明中有一关键性失误, 故[6]中的反例尚不能说明上述结论成立. 本文还对具有 σ-CF 拟基的 k 空间给出一个刻画定理.

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On Hyperspace of Lašnev Space*

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Abstract In this paper, some properties of the hyperspaces of nonempty compact subsets of some Lašnev spaces are discussed and a mistake in [6] is corrected.

Keywords closure-preserving, CF family, Lašnev space, stratifiable space.

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0. Introduction

In [6] T. Mizokami wanted to show that there exists a countable Lašnev space X such that its hyperspace $\mathbf{K}(X)$ of compact subsets is not stratifiable space. However, in the proof given by T. Mizokami there is a critical mistake, hence this problem is still unsolved.

In Section 1 we will construct a Lašnev space which is different from that given by T. Mizokami and prove that its hyperspace of compact subsets is not stratifiable.

In Section 2, a property of Lašnev space with σ-compact finite close K-network is characterized, and an example is given to show that there exists a Lašnev space which has no σ-compact finite closed K-network and its hyperspace is M_0 space.

Finally, we prove that a k'-space which has a σ-CF quasi-base is metrizable space.

Every space in this paper is assumed to be regular Hausdorff space. Let ω denote the set of all positive integers and ω_1 the first uncountable order number. Other notations in general topology are referred to [2]. The extent of space X is denoted by $e(X)$. The symbol Q denotes the set of rational numbers

Let $\mathbf{K}(X)$ denote the hyperspace of nonempty compact subsets of X with finite topology. $\mathbf{F}(X)(\subset \mathbf{K}(X))$ is the space of finite subsets of X. Other relative notions and notations of hyperspace can be found in [5].

1. An Example

The closed continuous image of metrizable space is called Lašnev space. M_2 (or M_3) space is

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called stratifiable space. The problem that whether the hyperspace of a Lašnev space is stratifiable is still open ([6], [8], [7]).

Remark 1 In [6], T. Mizokami wants to show that there is a Lašnev space X such that $\mathbf{K}(X)$ is not stratifiable (example 2.1). Unfortunately, his proof has a critical mistake which can not be made up. However, Mizokami's idea still greatly inspired us.

Example 1.1 There exists a countable Lašnev space X such that $\mathbf{K}(X)$ is not stratifiable space.

(i) **The construction of** X. Let S be the set of irrational numbers in crosed interval $[0, 1]$; D be a countable dense subsets of S; $N_0 = \{0\} \cup \{1/n: n \in \omega\}$.

Let $X' \triangleq D \times N_0 \cup S \times \{0\} \subset R^2$. $X = X'/S \times \{0\}$ is the quotient space obtained by identifying $S \times \{0\}$ to a point. $f: X' \to X$ denotes the quotient mapping and $p = f(S \times \{0\})$. Obviously f is a closed mapping, hence X is a Lašnev space.

Remark 2 The space X in [6] is constructed as follows:

$X' \triangleq \{x \in Q: 0 \leq x \leq 1 \text{ and } x \neq 1/n (n \in \omega)\} \times (\{0\} \cup \{1/n: n \in \omega\})$;

$A = \{(x, 0): (x, 0) \in X'\}$;

$X = X'/A$, $p = f(A)$;

$N'_k = (1/(k+1), 1/k) \times [0, 1/k] \cap X'$; $N' \triangleq \cup_{k \in \omega} N'_k$;

$N = f(N')$, where $(1/(k+1), 1/k)$ is open interval.

For comparison, in the following most of the notations are adopted from [6], but some will be regulated.

(ii) **The proof that** $\mathbf{K}(X)$ is not stratifiable space. Suppose that $\mathbf{K}(X)$ is stratifiable space, then there exists a CP closed neighbourhood base $\mathcal{B}$ of $\{p\}$ in $\mathbf{K}(X)$ ([1] Lemma 7.3). For each $\hat{B} \in \mathcal{B}$ denote $O(\hat{B}) = \cup\{F \in \mathbf{F}(X); p \in F \in \hat{B}\}$, it is easy to know that $O(\mathcal{B}) = \{O(\hat{B}); \hat{B} \in \mathcal{B}\}$ is a neighbourhood base of p in X. Other notations are as follows:

For every $d \in D$,

$$I'_n(d) = (\{d\} \times (0, 1/n]) \cap X'; \quad I_n(d) = f[I'_n(d)];$$

for $x_i \in I(d_i)$ $(i \leq n)$ (where $I(d) = I_1(d)$),

$$\mathcal{B}(x_1, \cdots, x_n) = \{\hat{B} \in \mathcal{B}: \{p, x_1, \cdots, x_n\} \in \mathrm{Int}\,\hat{B}\};$$

for $r_1, r_2 \in Q \cap [0, 1]$, $r_1 < r_2$;

$$S'(r_1, r_2) = \{(a, b) \in X': r_1 < a < r_2\} = ((r_1, r_2) \times N_0) \cap X';$$
$$S(r_1, r_2) = f[S'(r_1, r_2)];$$
$$O(\mathcal{B})/S(r_1, r_2) = \{O(\hat{B}) \cap S(r_1, r_2): \hat{B} \in \mathcal{B}\}.$$

Firstly, we will prove the following basic fact. If $\mathcal{B}' \subset \mathcal{B}$, and $O(\mathcal{B}')$ restricted in $S(r_1, r_2)$ is a neighbourhood base of p in $S(r_1, r_2)$, then for every $d \in D \cap (r_1, r_2)$, every $\beta_1, \beta_2 \in (r_1, r_2) \cap Q$ $(\beta_1 < \beta_2)$ and $n \in \omega$, there exist $x \in I_n(d)$ and two rational numbers $\alpha_1, \alpha_2 \in (\beta_1, \beta_2)$ such that $O(\mathcal{B}'$

$(x))/S(\alpha_1, \alpha_2)$ is a neighbourhood base of p in $S(\alpha_1, \alpha_2)$.

If otherwise, then there exist $d \in D \cap (r_1, r_2)$, $\beta_1, \beta_2 \in (r_1, r_2) \cap Q$, for every $x_i \in I(d)$ and α_i, $\alpha_{i+1} \in (\beta_1, \beta_2) \cap Q$ $(\alpha_i < \alpha_{i+1})$, there exists a neighbourhood V_i of p in $S(\alpha_i, \alpha_{i+1})$ such that $S(\alpha_i, \alpha_{i+1}) \cap O(\hat{B}) \not\subset V_i$ for every $\hat{B} \in \mathcal{B}'(x_i)$. Since $I(d)$ is a countable set denote $I(d) = \{x_i: i \in \omega\}$. Take a strictly increasing rational number sequence $\{\alpha_i\}_{i\in\omega}$, such that $\alpha_1 = \beta_1$, $\lim \alpha_i = \beta_2$. For each x_i and α_i, α_{i+1}, take V_i as above narration. Let $V = \bigcup_{i\in\omega} V_i$, then V is a neighbourhood of p in $S(\beta_1, \beta_2)$. Since $O(\mathcal{B}')/S(\beta_1, \beta_2)$ is also a neighbourhood base of p in $S(\beta_1, \beta_2)$ and one will easily know that

$$O(\mathcal{B}') = \bigcup_{i\in\omega} O(\mathcal{B}'(x_i)),$$

so there exists $j \in \omega$ and $\hat{B} \in \mathcal{B}'(x_j)$ such that

$O(\hat{B}) \cap S(\beta_1, \beta_2) \subset V$, therefore $O(\hat{B}) \cap S(\alpha_j, \alpha_{j+1}) \subset V_j$, this is a contradiction.

Let $\{b_i\}_{i\in\omega}$ denote the set of rational numbers in $(0, 1)$. Applying above proved facts, we take inductively countable many elements $\hat{B}_n (n \in \omega)$ in $\mathbf{B}$ and a point sequence $\{x_n\}_{n\in\omega}$.

(i) Choose $d_1 \in D$. We take $x_1 \in I(d_1)$ and rational number $\alpha_1 < \beta_1$ such that $O(\mathcal{B}(x_1))/S(\alpha_1, \beta_1)$ is a neighbourhood base of p in $S(\alpha_1, \beta_1)$ and $b_1 \notin (\alpha_1, \alpha_1)$.

Take $d_2 \in D \cap (\alpha_1, \beta_1)$, $x_2 \in I_2(d_2)$, $\alpha_2, \beta_2 \in (\alpha_1, \beta_1) \cap Q$ such that $b_2 \notin (\alpha_2, \beta_2)$ and $O(\mathcal{B}(x_1, x_2))/S(\alpha_2, \beta_2)$ is a neighbourhood base of p in $S(\alpha_2, \beta_2)$.

Since $O(\mathcal{B}(x_1))/S(\alpha_1, \beta_1)$ is a neighbourhood base of p in $S(\alpha_1, \beta_1)$, choose $\hat{B}_1 \in \mathcal{B}(x_1)$ such that $x_2 \notin O(\hat{B}_1)$.

(ii) For $n \geq 1$, assume that $d_n \in D \cap (\alpha_{n-1}, \beta_{n-1})$, $x_n \in I_n(d_n)$, $\alpha_n, \beta_n \in (\alpha_{n-1}, \beta_{n-1}) \cap Q$, $\hat{B}_{n-1} \in \mathcal{B}(x_1, \cdots, x_{n-1})$ have been chosen and satisfy:

(1) $O(\mathcal{B}(x_1, \cdots, x_n))/S(\alpha_n, \beta_n)$ is a neighbourhood base of p in $S(\alpha_n, \beta_n)$;

(2) $b_n \notin (\alpha_n, \beta_n)$;

(3) $x_n \notin O(\hat{B}_{n-1})$.

Then choose $d_{n+1} \in D \cap (\alpha_n, \beta_n)$, $x_{n+1} \in I_{n+1}(d_{n+1})$, $\alpha_{n+1}, \beta_{n+1} \in (\alpha_n, \beta_n) \cap Q$ such that $b_{n+1} \notin (\alpha_{n+1}, \beta_{n+1})$ and $O(B(x_1, \cdots, x_{n+1}))/S(\alpha_{n+1}, \beta_{n+1})$ is a neighbourhood base of p in $S(\alpha_{n+1}, \beta_{n+1})$. By inductive assumption we can choose $\hat{B}_n \in \mathcal{B}(x_1, \cdots, x_n)$ and $x_{n+1} \notin O(\hat{B}_n)$.

As above we take inductively $\{\hat{B}_i\}_{i\in\omega}$, $\{x_i\}_{i\in\omega}$ and interval family $\{(\alpha_n, \beta_n)\}_{n\in\omega}$ satisfying:

(a) $\lim \alpha_n = \lim \beta_n = y \in S$, and in X' $\lim x_i = (y, 0) \in X'$, then in X $\lim x_i = p$.

(b) $\{x_1, \cdots, x_n, p\} \in \hat{B}_n$, but $\{p\} \cup \{x_i\}_{i\in\omega} \notin \hat{B}_n$, this is due to $x_{n+1} \notin O(\hat{B}_n)$.

So the compact set $F = \{p\} \cup \{x_i\}_{i\in\omega} \notin \bigcup_{n\in\omega} \hat{B}_n$. Take arbitrarily a neighbourhood $\langle V_1, \cdots, V_m \rangle$ of F in $\mathbf{K}(X)$, where V_i is an open subset of X. It is readily to verify that there exists k such that $\{x_1, \cdots, x_k, p\} \in \langle V_1, \cdots, V_n \rangle$, then $\langle V_1, \cdots, V_m \rangle \cap \hat{B}_k \neq \emptyset$. Hence $F \in CL(\bigcup_{n\in\omega} B_n)$. Howover, $\mathcal{B}$ is a CP closed neighbourhood base, this is a contradiction.

Remark 3 There are some misprints in [6], but the key fault is that the set $\langle N \rangle$ is regarded as a neighbourhood of $\{p\}$ in $\mathbf{K}(X)$. In fact, N is not even a neighbourhood of p in X. In addition, if the concrete construction of neighbourhood base of p is not given defintely, the results that the neighbourhood base of $\{p\}$ is not CP family can hardly be proved. Moreover, even if $\mathbf{K}(X)$ of Example 2.1 in [6] is not M_2 space, the proof will also be more difficult than that in this paper.

2 K-network, CF quasi-base and Lašnev space

Let $\mathbf{U}$ be a family of subsets of X. We call $\mathbf{U}$ compact-finite if $\{U: U\in\mathbf{U}, U\cap F\neq\emptyset\}$ is finite for each $F\in\mathbf{K}(X)$; $\mathbf{U}$ is a K-network of X if for each $K\in\mathbf{K}(X)$ and every open set $V\supset K$ there exists a finite family $\mathbf{U}_0\subset\mathbf{U}$ such that $K\subset\cup\mathbf{U}_0\subset V$; $\mathbf{U}$ is CF family if $\mathbf{U}/K$ is finite family for every $K\in\mathbf{K}(X)$; $\mathbf{U}$ is a quasi-base of X if for every open subset V of X and every point $x\in V$, there exists $U\in\mathbf{U}$ such that $x\in \mathrm{Int}U\subset U\subset V$.

It is known that compact-finite family must be CF family.

The following results are well-known.

Theorem A *For a space X the following are equivalent:*

(1) *X is a Lašnev space.*

(2) *X is a Fréchet space and has a σ-compact finite K-network.*

(3) *X is a Fréchet space and has a σ-CF K-network.*

From (2), it is natural to ask that whether every Lašnev space has a σ-compact-finite closed K-network[6]. Y. Ge and C. Liu have proved that a Lašnev space has a σ-compact-finite closed K-network if and only if it is a N space[3],[4]. Here, we give a necessary condition for the existence of the σ-compact-finite closed K-network of any Lašnev space.

Theorem 2.1 *X is a space and has a σ-compct-finite closed K-network. For every metrizable space M, if there exists a continuous closed mapping $f: M\to X$, then for every $x\in X$, the inequality $e(f^{-1}[x]\cap \mathrm{CL}(M\setminus f^{-1}[x]))\le \aleph_0$ holds.*

The proof is directed and so is omitted.

In [6], T. Mizokami proved that if a Lašnev space X has a σ-compact-finite closed K-network, then $\mathbf{K}(X)$ is paracompact σ space.

The following example shows that this condition is not necessary.

Example 2.2 There exists a Lašnev space X which has no σ-compact-finite closed K-network and $\mathbf{K}(X)$ is M_0 space.

A space is called M_0 space if it has a σ-CP base consisting of clopen sets.

(1) **The construction of** X. Denote $M=\oplus\{M_\alpha: \alpha<\omega_1\}$, where $M_\alpha=\{x_n^\alpha: n\in\omega\}\cup\{x_\omega^\alpha\}$ is homeomorphic copy of $N_0=\{0\}\cup\{1/n: n\in\omega\}$, where x_ω^α is the unique accumulation point of M_α; let $A=\{x_\omega^\alpha: \alpha<\omega_1\}$, $X=M/A$ is the quotient space obtained by identifying A to a point. Let $f: M\to X$ be the quotient mapping and $p=f(A)$. Since $e(f^{-1}(p)\cap \mathrm{CL}(M\setminus f^{-1}(p)))=\aleph_1$, from theorem 2.1 or [3], X has no σ-compact-finite closed K-network.

(2) **The proof that** $\mathbf{K}(X)$ is M_0 space. We still denote $f(x_i^\alpha)=x_i^\alpha (i<\omega)$; $X_k=\{x_i^\alpha: i\le k, \alpha<$

$\omega\}$; $\mathbf{U}(p)=\{U: p\in U$; for each $\alpha<\omega$ there exists $k\in\omega$ such that $U\cap I_\alpha=I_\alpha^k\}$, where $I_\alpha=f[M_\alpha]$, $I_\alpha^k=\{x_i^\alpha: i\geq k\}$. Obviously $\mathbf{U}(P)$ is a clopen neighbourhood base of p. Define $\mathbf{F}_k=\hat{\mathbf{F}}(X_k)$; for each $F=\{x_1,\cdots,x_n\}\in\mathbf{F}(X\setminus\{p\})$, and $U\in\mathbf{U}(p)$ with $U\cap F=\emptyset$, we denote $\hat{F}=\langle\{x_1,\cdots,x_n\}\rangle$ and $\hat{U}(F)=\langle U,\{x_1\},\cdots,\{x_n\}\rangle$

$$\mathbf{U}_k=\{\hat{U}(F): F\in\mathbf{F}_k; U\in\mathbf{U}(p)\ and\ U\cap X_k=\emptyset\},\ \{\mathbf{H}_k=\hat{F}: F\in\{\mathbf{F}_k\}.$$

To begin with note that $\{\cup_{k\in\omega}\mathbf{H}_k\}\cup\{\cup_{k\in\omega}\mathbf{U}_k\}$ is a topology base of $\mathbf{K}(X)$. Let $E\in\langle V_1,\cdots,V_m\rangle$; where $V_1,\cdots,V_m$ are open subsets of X. It suffices to consider the situation where E is a infinite compact subset. Since p is the unique accumulation point in X, $p\in E$, without loss of generality assume $p\in\cap\{V_i, i\leq k_0\}$ and $p\notin\cup\{V_i: k_0<i\leq m\}$, take $V\in\mathbf{U}(p)$ and $V\subset\cap_{i<k}V_i$, then $E\setminus V$ is finite. Take $a_i\in E\cap V_{k+i}(i\leq m-k_0)$, $A=\{a_i: i\leq m-k_0\}$, there exists n such that $X_n\supset(E\setminus V)\cup A$. Denote $F=(V\cap E\cap X_n)\cup A\cup(E\setminus V)$ and $U=V\setminus X_n$. Since $V\cap E\cap X_n$ is finite, it is easy to see that $E\in\hat{U}(F)\in\mathbf{U}_n$ and $\hat{U}(F)\subset\langle V_1,\cdots,V_m\rangle$.

One can readily check that $\mathbf{H}_k$ is a CP family. Now, we show that $\mathbf{U}_k$ is a CP family for each $k\in\omega$. Let $\mathbf{U}'\subset\mathbf{U}_k$, $E\notin\cup\mathbf{U}'$. Since E is compact. $E\cap X_k$ and $A=\{\alpha: E\cap(I_\alpha^{k+1}\setminus\{p\})\neq\emptyset$ are finite sets. Denote $A=\{\alpha_i\}_{i<n}$. For each $\alpha_i\in A$, take $m_i=\min\{j: x_j^{\alpha_i}\in E\cap(I_{\alpha_i}^{k+1}\setminus\{p\})\}$; $F_0=(E\cap X_k)\cup(x_{m_i}^{\alpha_i}: i\leq n\}$; $V=X\setminus F_0$. Without loss of generality, assume that $|E|=N_0$. Then $\hat{V}(F_0)$ is a neighbourhood of E in $\mathbf{K}(X)$. For any $\hat{U}(F)\in\mathbf{U}_k$, if $K\in\hat{U}(F)\cap\hat{V}(F_0)\neq\emptyset$, then $E\cap X_k=K\cap X_k=F$, and $\{x_{m_i}^{\alpha_i}\}_{i\leq n}\subset K\cap(X\setminus X_k)\subset U$, so $E\cap(X\setminus X_k)\subset U$, thus $E\in\hat{U}(F)$. Since $E\notin\cup\mathbf{U}'$, we have $\hat{V}(F_0)\cap(\cup\mathbf{U}')=\emptyset$, hence $\cup\mathbf{U}'$ is closed. This completes the proof.

In [6], T. Mizodami gave an example to show that there exists a Lašnev space without σ-CF quasi-base and raised the question; "Which kinds of spaces have a σ-CF quasi-base?" The following results characterize completely those k'-space (thus Fréchet spaces and Lašnev spaces) which have such a base.

Theorem 2.3 *k'-space with σ-CF quasi-base is metrizable space.*

Proof Let $\mathcal{B}=\cup\{\mathcal{B}_n: n\in\omega\}$ be a quasi-base of k'-space X, where $\mathcal{B}_n$ is CF family. For each $x\in X$, denote $\mathcal{B}_n(x)=\{B\in\mathcal{B}_n: x\in \mathrm{Int}B\}$; $V_n(x)=\cap\mathcal{B}_n(x)$. Since $\mathcal{B}$ is a quasi-base, for every U containing x there exists $n\in\omega$ such that $V_n(x)\subset U$. We prove that $V_n(x)$ is a neighbourhood of x. If not, then $x\in \mathrm{CL}(X\setminus V_n(x))$. Since X is k'-space, there exists a compact subset F such that $x\in CL(X\setminus V_n(x))\cap F)\subset F$. Denote the compact subset $\mathrm{CL}(F\cap(X\setminus V_n(x))=H$. $\{B\cap H: B\in\mathcal{B}_n(x)\}$ is a finite family. Let $B_i\in\mathcal{B}_n(x)$ $(i\leq m)$ such that $\{B_i\cap H: i\leq m\}=\{B\cap H: B\in\mathcal{B}_n(x)\}$. Then

$$(\bigcap_{i\leq m}B_i)\cap H=\cap\{B\cap H: B\in\mathcal{B}_n(x)\}=V_n(x)\cap H.$$

$$H\setminus(\bigcap_{i\leq m}B_i)=H\setminus V_n(x)\supset(X\setminus V_n(x))\cap F.$$

Since $x\in \mathrm{Int}B$ $(i\leq m)$ and $\cap_{i\leq m}B_i$ is a neighbourhood of x,

$$x\notin \mathrm{CL}(H\setminus\bigcap_{i\leq m}B_i)\supset \mathrm{CL}(F\cap(X\setminus V_n(x)).$$