Journal of Mathematical Research & Exposition
Vol. 17, No. 3, 319-326, Aug. 1997

The Generalization of Whitney's Lemma and Application*

Cen Yanbin
(Dept. of Math., Qiannan Teachers' College, Guizhou Duyun 558000)

Cen Yanming
(Dept. of Math., Guizhou Institute for Nationalities, Guiyang 550025)

Abstract Whitney's lemma is an important theorem in the local singularity theory of germs of C^∞ functions. In this paper, we prove the global conclusion of this lemma. Based on this generalization, the plastic yield criterion for certain kind materials is discussed in detail. We find that for this kind materials the most general form of the plastic yield criterion should be $g(J, J_2', J_3'^2) = 0$. Finally, we shall also give some practical examples for explanation.

Key words Whitney's lemma, generalization, yield criterion.

Classification AMS(1991) 58C27, 32S20, 73E05/CCL O186.33, O344.1

1. Preliminaries and symbols

Notation: Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \cdots, \beta_k)$, $(\alpha, \beta) = (\alpha_1, \alpha_2, \cdots, \alpha_n, \beta_1, \beta_2, \beta_k)$, α_i $(i = 1, 2, \cdots, n)$ and β_j $(j = 1, 2, \cdots, k)$ be non-negative integers. Then

$$|\alpha| = \sum_{i=1}^{n} \alpha_i, \quad |\beta| = \sum_{j=1}^{k} \beta_j,$$

$$D^{\alpha,\beta} f(x, y) = \frac{\partial^{|\alpha|+|\beta|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n} \partial y_1^{\beta_1} \cdots \partial y_k^{\beta_k}} f(x, y),$$

where $(x, y) = (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_k) \in \mathbf{R}^n \times \mathbf{R}^k$.

The ring of C^∞ functions on $\mathbf{R}^n \times \mathbf{R}^k$ will be denoted by $C^\infty(n+k)$. $M(k) = \{f \in C^\infty(n+k) \mid f|_{\mathbf{R}^n \times \{0\}} = 0\}$, then $M(k)$ is an idea in $C^\infty(n+k)$. It is easy to show that $f \in M(k)$ if and only if

$$f = \sum_{j=1}^{k} f_j y_j, \; f_j \in C^\infty(n+k).$$

---

* Received Sep. 14, 1994.

$$M(k)^s = \{f \in C^\infty(n+k) \mid D^{\alpha,\beta} f \mid_{\mathbf{R}^n\times\{0\}} = 0 \text{ for all } \alpha \text{ and } \beta \text{ with } |\beta| < s\}.$$

$\mathbf{R}^n \times \{0\}$ is a subspace of $\mathbf{R}^n \times \mathbf{R}^k$: $\{(x,y) \in \mathbf{R}^n \times \mathbf{R}^k \mid_{y_1=y_2=\cdots=y_k=0}\}$.

Definition 1 *A* C^∞ *function* f *on* $\mathbf{R}^n \times \mathbf{R}^k$ *is called flat on the subspace* $\mathbf{R}^n \times \{0\} \subset \mathbf{R}^n \times \mathbf{R}^k$, *if* f *and its partial derivatives of all orders are zero at every point of* $\mathbf{R}^n \times \{0\}$.

Definition 2 $M(k)^\infty = \bigcap_{S=1}^{\infty} M(k)^s$.

It is obvious that $M(k)^\infty$ is the set of all flat functions on $\mathbf{R}^n \times \{0\}$.

Henceforth, $M(1)^\infty$ is the focal point of our work, i.e., the flat functions on $\mathbf{R}^n \times \{0\} \subset \mathbf{R}^n \times \mathbf{R}$.

2 Lemmas

Lemma 1 $\forall s \in N$, *the following equality holds*: $M(k)^s \cdot M(k)^\infty = M(k)^\infty$.

Proof First, we will show that $M(k) \cdot M(k)^\infty = M(k)^\infty$.

(1) $\forall f \in M(k) \subset C^\infty(n+k)$, $g \in M(k)^\infty$, because $M(k)^\infty$ is an ideal of $C^\infty(n+k)$, therefore $f \cdot g \in M(k)^\infty \Rightarrow M(k) \cdot M(k)^\infty \subseteq M(k)^\infty$.

(2) $\forall g \in M(k)^\infty$,

$$g(x,y) = \int_0^1 \frac{d}{dt} g(x,ty)\,dt = \int_0^1 \left(\sum_{j=1}^{k} \frac{\partial}{\partial y_j} g(x,ty) y_j \right) dt = \sum_{j=1}^{k} h_j(x,y) y_j,$$

where

$$h_j(x,y) = \int_0^1 \frac{\partial}{\partial y_j} g(x,ty)\,dt$$

We need only to show that $h_j \in M(k)^\infty$ $(j = 1, 2, \cdots, k)$.

$\forall \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \cdots, \beta_k)$, we have

$$\begin{aligned} D^{\alpha,\beta} h_j(x,0) &= D^{\alpha,\beta} \left(\int_0^1 \frac{\partial}{\partial y_j} g(x,ty)\,dt \right) \Bigg|_{y_1=0,\cdots,y_k=0} \\ &= \left(\int_0^1 t^{|\beta|} \frac{\partial^{|\alpha|+|\beta|+1}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n} \partial y_1^{\beta_1} \cdots \partial y_j^{\beta_j+1} \cdots \partial y_k^{\beta_k}} g(x,ty)\,dt \right) \Bigg|_{y_1=0,\cdots,y_k=0} \end{aligned}$$

Note that if $g \in M(k)^\infty$, then its partial derivatives of all orders belong to $M(k)^\infty$, too (Definition of the flat function). From this,

$$\frac{\partial^{|\alpha|+|\beta|+1}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n} \partial y_1^{\beta_1} \cdots \partial y_j^{\beta_j+1} \cdots \partial y_k^{\beta_k}} g(x,0) = 0.$$

Therefore, $D^{\alpha,\beta} h_j(x,0) = 0 \Rightarrow h_j \in M(k)^\infty$. Furthermore, $M(k)^\infty \subseteq M(k) M(k)^\infty$.

Synthesize (1) and (2): $M(k) M(k)^\infty = M(k)^\infty$.

Using the above equality and by induction, we know that $\forall s \in N$,

$$M(k)^s M(k)^\infty = M(k)^\infty.$$

Lemma 2 *$f \in M(k)^\infty$ if and only if $\forall s \in N$, f can be represented by the following form:*

$$f(x, y) = \sum_{|\beta| = s} f_\beta(x, y) y_1^{\beta_1} \cdots y_k^{\beta_k},$$

where $f_\beta \in M(k)^\infty$.

Proof It can be directly deduced from Lemma 1.

Lemma 3 (Generalization of E. Boerel Theorem) *Given a sequence of C^∞ functions on* $\mathbf{R}^n$ $\{f_m(x)\}$ $(m = 0, 1, 2, \cdots)$, *there exists a C^∞ function on $\mathbf{R}^n \times \mathbf{R}$* $f: \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ such that

$$D^{0,m} f(x, y) \big|_{\mathbf{R}^n \times \{0\}} = f_m(x).$$

Proof $\forall r > 0$ and $f_m(x)$, we can make a C^∞ function $\overline{f}_m$ with compact support $K(0, r)$ which is the closed ball around 0 with radius r: $|x| \le r$, and $\overline{f}_m(x) = f_m(x)$ on $K(0, \frac{r}{2})$. For simple and convenient, $\overline{f}_m(x)$ is written as $f_m(x)$ yet. Once more take a C^∞ function

$$\varphi: R \to R, 0 \le \varphi \le 1, \varphi(y) = 1 \text{ for all } |y| \le \frac{r}{2}, \varphi(y) = 0 \text{ for } |y| \ge r.$$

Let

$$f(x, y) = \sum_{m=0}^{\infty} \frac{f_m(x)}{m!} y^m \varphi(t_m y). \tag{1}$$

Assume that the sequence t_m $(m = 0, 1, 2, \cdots)$ can be defined to make the series

$$\sum_{m=0}^{\infty} D^{\alpha, p} \left(\frac{f_m(x)}{m!} y^m \varphi(t_m y) \right) \tag{2}$$

convergnet uniformly for each $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and each natural number p, then $f(x, y)$ would be a C^∞ function and could be differentiated term by term.

Since for each t_m, $\varphi(t_m y) = 1$, as long as $|y|$ is small enough, and the differentiation is a local property. This gives

$$D^{0,m} f(x, y) \big|_{\mathbf{R}^n \times \{0\}} = f_m(x).$$

Therefore we need to show that a sufficiently rapidly increasing sequence $\{t_m\}$ makes the series (2) uniformly convergent for every α. Now, write the m-th term in (1) in the form:

$$\left(\frac{1}{t_m} \right)^m \frac{f_m(x)}{m!} (t_m y)^m \varphi(t_m y) = \left(\frac{1}{t_m} \right)^m f_m(x) \psi_m(t_m y).$$

The function ψ_m vanish outside $\{|t_m y| \le 1\}$.

Let $M_m = \max\{|D^{\alpha,p}(f_m(x)\Psi_m(y))|_{|\alpha|+p<m}\}$. Note that for each given m, there are only finitely many (α, p) with $|\alpha|+p<m$ and that $\operatorname{supp}(f_m\Psi_m)\subset\{(x,y)\,|\,|x|\leq r, |y|\leq r\}$. Hence M_m exists Since $t_m>1$, it follows that for $|\alpha|+p<m$ that

$$\left|D^{\alpha,p}\left(\frac{f_m(x)}{m!}y^m\varphi(t_my)\right)\right| \leq (t_m)^{|\alpha|+p}\left(\frac{1}{t_m}\right)^m M_m < \frac{M_m}{t_m}.$$

Now, choose a sequence $\epsilon_m>0$ such that $\sum_{m=0}^{\infty}\epsilon_m$ converges and choose $t_m>\frac{M_m}{\epsilon_m}$, then $\frac{M_m}{t_m}<\epsilon_m$. Thus, for $|\alpha|+p<m$, the m-th element of (2) is dominated by ϵ_m.

Since r is arbitrary, the lemma is true.

From this lemma, we know that for any given formal power series (not necessarily convergent) $\sum_{m=0}^{\infty}\frac{f_m(x)}{m!}y^m$, there exists a C^∞ function on $\mathbf{R}^n\times\mathbf{R}$ whose Taylor series at every point $(x,0)\in\mathbf{R}^n\times\{0\}$ is exactly this formal power series

3 The main results

Theorem 1 *Assume that* h: $\mathbf{R}^n\times\mathbf{R}\to R$ *is a* C^∞ *function and flat on* $\mathbf{R}^n\times\{0\}$ $((x,y)=(x_1,x_2,\cdots,x_n,y))$, *then*

$$r(x,y)=\begin{cases} h(x,\sqrt{y}) & \text{on } \mathbf{R}^n\times\mathbf{R}^+ \\ 0 & \text{on } \mathbf{R}^n\times\{0\} \\ h(x,-\sqrt{-y}) & \text{on } \mathbf{R}^n\times\mathbf{R}^- \end{cases}$$

is C^∞ *on* $\mathbf{R}^n\times\mathbf{R}$ *and flat on* $\mathbf{R}^n\times\{0\}$.

Proof Note that

$$\varphi_1: \mathbf{R}^n\times\mathbf{R}^+\to\mathbf{R}^n\times\mathbf{R}^+,\ (x,y)\mapsto(x,\sqrt{y})$$

and

$$\varphi_2: \mathbf{R}^n\times\mathbf{R}^-\to\mathbf{R}^n\times\mathbf{R}^-,\ (x,y)\mapsto(x,-\sqrt{-y})$$

are an entirety diffemorphism (C^∞ topology map) from $\mathbf{R}^n\times R^+$ onto $\mathbf{R}^n\times R^+$ and from $\mathbf{R}^n\times\mathbf{R}^-$ onto $\mathbf{R}^n\times\mathbf{R}^-$, respectively, and $h(x,y)$ is a C^∞ function on $\mathbf{R}^n\times\mathbf{R}^+$ and on $\mathbf{R}^n\times\mathbf{R}^-$, respectively.

The key point of the proof is to show that $r(x,y)$ is flat on $\mathbf{R}^n\times\{0\}$, i.e., $r(x,y)$ and its partial derivatives of all orders are continuous and zero at every point of $\mathbf{R}^n\times\{0\}$. Consider

case (1) As $y\to+0$:

Since $h(x,y)$ is differentiable on $\mathbf{R}^n\times\mathbf{R}$ and flat on $\mathbf{R}^n\times\{0\}$, therefore, for any $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$, $\alpha_i\geq 0$ $(i=1,2,\cdots,n)$, one knows easily that

$$D_+^{\alpha,0}r(x,0)=0.$$

Now, by induction we will prove that for each $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$ and any positive integer β,

$$D_+^{\alpha,\beta} r(x,0) = 0.$$

As $\beta = 1$,

$$D_+^{\alpha,1} r(x,0) = \lim_{\triangle y \to +0} \frac{D^{\alpha,0} h(x, \triangle y) - D^{\alpha,0} h(x,0)}{\triangle y} = \lim_{\triangle y \to +0} \frac{D^{\alpha,0} h(x, \triangle y)}{\triangle y}.$$

However, $D^{\alpha,0} h(x,y)$ is also flat on $\mathbf{R}^n \times \{0\}$. Therefore, $\forall s \in N$, there exists a flat function $\Psi(x,y)$ on $\mathbf{R}^n \times \{0\}$ such that

$$D^{\alpha,0} h(x, \sqrt{\Delta y}) = (\Delta y)^{\frac{s}{2}} \Psi(x, \sqrt{\Delta y}).$$

Take $s \geq 3$, we have

$$D^{\alpha,1} r(x,0) = \lim_{\Delta y \to +0} \frac{(\Delta y)^{\frac{s}{2}} \Psi(x, \sqrt{\Delta y})}{\Delta y} = 0.$$

Assume that $D_+^{\alpha,k} r(x,0) = 0$ holds for $\beta = k$.

As $\beta = k+1$, from the expression of $r(x,y)$, by induction, we obtain that as $y > 0$, $\forall k \in N$, $D^{\alpha,k} r(x,y)$ has the following form:

$$D^{\alpha,k} r(x,y) = \frac{a_k D_u^{\alpha,k} h(x, \sqrt{y})}{(\sqrt{y})^k} + \frac{a_{k+1} D_u^{\alpha,k+1} h(x, \sqrt{y})}{(\sqrt{y})^{k+1}} + \cdots + \frac{a_{2k-1} D_u^{\alpha,2k-1} h(x, \sqrt{y})}{(\sqrt{y})^{2k-1}}, \quad (3)$$

where $a_k, a_{k+1}, \cdots, a_{2k-1}$ are constants (independent of x and y), $u = \sqrt{y}$.

In order to show that $D_+^{\alpha,k+1} r(x,0) = 0$, we need only to show that for each i ($i = 1, 2, \cdots, k-1$), the right derivative of $G_i(x,y)$ with respect to y at $y = 0$ is zero, i.e. $D_+^{0,1} G_i(x,0) = 0$. Where,

$$G_i(x,y) = \begin{cases} \dfrac{a_{k+i} D_u^{\alpha,k+i} h(x, \sqrt{y})}{(\sqrt{y})^{k+i}}, & (x,y) \in \mathbf{R}^n \times \mathbf{R}^+, \\ 0, & (x,y) \in \mathbf{R}^n \times \{0\}. \end{cases} \quad (4)$$

Note that $D_u^{\alpha,k+i} h(x,u)$ is flat on $\mathbf{R}^n \times \{0\}$, therefore $\forall s \in N$, there exists the flat function on $\mathbf{R}^n \times \{0\}$ $g_{k+i}(x,u)$ such that $D_u^{\alpha,k+i} h(x, \sqrt{y}) = y^{\frac{s}{2}} g_{k+i}(x, \sqrt{y})$.

Using the above expression and according to definition: $D_+^{0,1} G_i(x,0) = 0$.

case (2) As $y \to -0$, similar to case (1), we know that $D_-^{\alpha,\beta} r(x,0) = 0$.

Synthesize case (1) and (2): $D^{\alpha,\beta} r(x,0) = 0$, i.e. $r(x,y)$ and its partial derivatives of all orders are zero on $\mathbf{R}^n \times \{0\}$.

Finally, using lemma 2 and the expressions of $r(x,y)$ and $D^{\alpha,\beta} r(x,y)$, we have that for any $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and for any $s \in N$ and any non-negative integer β, there exists the flat function $\epsilon_{\alpha,\beta,s}(x,u)$ on $\mathbf{R}^n \times \{0\}$ such that

$$|D^{\alpha,\beta} r(x,y)| \leq |y|^{\frac{s}{2}} \, \epsilon_{\alpha,\beta,s}(x, \sqrt{y}).$$

It follows that

$$\lim_{(x,y) \to (x_0,0)} D^{\alpha,\beta} r(x,y) = 0 = D^{\alpha,\beta}(x_0, 0).$$

Therefore, $r(x, y)$ and its partial derivatives of all orders are continuous at every point of $\mathbf{R}^n \times \{0\}$. Thereby, $r(x, y)$ is C^∞ and flat on $\mathbf{R}^n \times \{0\}$.

Theorem 2 (Global Whitney's Lemma) *Let $f(x, y)$ be a C^∞ function on $\mathbf{R}^n \times \mathbf{R}$, and $\forall (x, y) \in \mathbf{R}^n \times \mathbf{R}, f(x, -y) = f(x, y)$. Then there exists a C^∞ function*

$$g: \mathbf{R}^n \times (R^+ \cup \{0\}) \to R$$

such that $g(x, y^2) = f(x, y)$.

Proof Since $\forall (x, y) \in \mathbf{R}^n \times \mathbf{R}, f(x, -y) = f(x, y)$, therefore $f(x, y) = f(x, |y|)$.

Set $u = y^2 (\forall y \in R)$. Then $u \geq 0$ and $|y| = \sqrt{u}$. Thus

$$f(x, y) = f(x, |y|) = f(x, \sqrt{u}).$$

Because $\varphi: \mathbf{R}^n \times \mathbf{R}^+ \to \mathbf{R}^n \times \mathbf{R}^+, (x, u) \mapsto (x, \sqrt{u})$ is an entirety diffemorphism from $\mathbf{R}^n \times \mathbf{R}^+$ onto $\mathbf{R}^n \times \mathbf{R}^+$, therefore $f(x, \sqrt{u})$ is C^∞ on $\mathbf{R}^n \times \mathbf{R}^+$.

As $(x, u) \in \mathbf{R}^n \times \mathbf{R}^+$, choose $g(x, u)$, then $g(x, u) = f(x, \sqrt{u}) \in C^\infty$ on $\mathbf{R}^n \times \mathbf{R}^+$ and $g(x, y^2) = f(x, |y|) = f(x, y)$.

In a geneneral way, although $f(x, u)$ is C^∞ on $\mathbf{R}^n \times (R^+ \cup \{0\})$, as above stated, $f(x, \sqrt{u})$ is not necessarily C^∞ on $\mathbf{R}^n \times \{0\}$! However, under the hypothesis of this theorem, we can prove that $f(x, \sqrt{u})$ and its partial derivatives of all orders can be extented continuously to $\mathbf{R}^n \times (R^+ \cup \{0\})$. Thus, $g(x, u) = f(x, \sqrt{u})$ is also C^∞ on $\mathbf{R}^n \times (R^+ \cup \{0\})$.

Since $\forall (x, y) \in \mathbf{R}^n \times \mathbf{R}, f(x, -y) = f(x, y)$. Therefore, $D^{0, 2m-1} f(x, 0) = -D^{0, 2m-1} f(x, 0)$. Thereby $D^{0, 2m-1} f(x, 0) = 0$. Therefore, $f(x, y)$ has the formal power series at every point $(x, 0) \in \mathbf{R}^n \times \{0\})$ as follows:

$$\sum_{m=0}^{\infty} \frac{D^{0, 2m} f(x, 0)}{(2m)!} y^{2m}.$$

Consider the following formal power series:

$$\sum_{m=0}^{\infty} \frac{D^{0, 2m} f(x, 0)}{(2m)!} u^{m}.$$

From Lemma 3, we know that there exists a C^∞ function $g_\alpha(x, u)$ on $\mathbf{R}^n \times \mathbf{R}$ whose Taylor series at every point $(x, 0) \in \mathbf{R}^n \times \{0\}$ is exactly this formal power series

Set $h(x, y) = f(x, y) - g_\alpha(x, y^2)$. Then $h(x, y)$ has two properties as follows:

(1) The formal power series of $h(x, y)$ at any point of $\mathbf{R}^n \times \{0\}$ is zero, thereby $h(x, y)$ is flat on $\mathbf{R}^n \times \{0\}$.

(2) $\forall (x, y) \in \mathbf{R}^n \times \mathbf{R}$, $h(x, -y) = h(x, y)$.

Using property (1) and Theorem 1, we know that

$$r(x, y) = \begin{cases} h(x, \sqrt{u}) & (x, u) \in \mathbf{R}^n \times \mathbf{R}^+ \\ 0 & (x, u) \in \mathbf{R}^n \times \{0\} \end{cases}$$

is C^∞ on $\mathbf{R}^n \times (R^+ \cup \{0\}$ and flat on $\mathbf{R}^n \times \{0\}$. In other words, $h(x, \sqrt{u})$ and its partial derivatives of all orders can be extented continously to $\mathbf{R}^n \times (R^+ \cup \{0\})$. Thus, $h(x, \sqrt{u})$

and $g_\alpha(x, u)$ are all C^∞ functions on $\mathbf{R}^n \times (R^+ \cup \{0\})$. From this, we can deduce that $f(x, \sqrt{u})$ is C^∞ on $\mathbf{R}^n \times (R^+ \cup \{0\})$.

Take $g(x, u) = f(x, \sqrt{u})$, $(x, u) \in \mathbf{R}^n \times (R^+ \cup \{0\})$, then $g(x, u)$ is C^∞ on $\mathbf{R}^n \times (R^+ \cup \{0\})$ and $g(x, y^2) = f(x, |y|) = f(x, y)$, $\forall (x, y) \in \mathbf{R}^n \times \mathbf{R}\}$.

4 The application and examples

Lemma 4 (G. Glaeser) *Assume that $f(x_1, \cdots, x_n)$ is a C^∞ and symmetric function in variable $x_1, \cdots, x_n$ (i.e. for any replacement of n letters π, always hold $f(x_1, \cdots, x_n) = f(x_{\pi(1)}, \cdots, x_{\pi(n)})$), then there exists a C^∞ function $g(y_1, \cdots, y_n)$ such that $f = g \circ N$. Where N denotes the Newton mapping:*

$$N = \begin{cases} y_1 = \sum_{i=1}^{n} x_i, \\ y_2 = \sum_{1 \le i < j \le n} x_i x_j, \\ \vdots \\ y_n = \prod_{i=1}^{n} x_i. \end{cases}$$

Proof It can be seen in [1] p. 125, 3.4.

Theorem 3 *Assume that the material is isotropic, the plastic yield function is C^∞ and the yield stresses (absolute value) for compression and pulling are the same, then the plastic yield surface can be written by the form: $g(J_1, J_2', J_3'^2) = 0$, where $J_1 = \sigma_1 + \sigma_2 + \sigma_3$, J_2', J_3' are the second and third invariants of the deviatoric stress tensor.*

Proof Since the material is isotropic, therefore the plastic surface can be written by $F(\sigma_1, \sigma_2, \sigma_3) = 0$, where F is a symmetric function in σ_1, σ_2 and σ_3, σ_1, σ_2 and σ_3 denote the principal stresses

First, second and third invariants of stress tensor are

$$N_1 = \begin{cases} J_1 = \sigma_1 + \sigma_2 + \sigma_3, \\ J_2 = (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1), \\ J_3 = \sigma_1\sigma_2\sigma_3 \end{cases}$$

and F is C^∞ and symmetric in σ_1, σ_2 and σ_3, from Lemma 4, we know that there exists a C^∞ function f such that

$$F = f \circ N_1.$$

Namely the plastic yield criterion can be represented by $f(J_1, J_2, J_3) = 0$.

Furthermore, from the relation between stress and the deviatoric stress tensor in plasticity, we easily know that the plastic yield surface can be denoted by

$$f(J_1, J_2', J_3') = 0.$$

According to [2], p. 17, if for compresion and pulling the yield stresses of the material are same, then f must be an even function of J_3'. From Theorem 2, there exists a C^∞ function g such that $g(J_1, J_2', J_3'^2) = f(J_1, J_2', J_3')$.

If we do not consider the effect of hydrostatic pressure, the plastic yield surface for this kind material should have the following form:

$$g(J_2', J_3'^2) = 0.$$

[2], p. 45 pointed out that Prager found that the abservations can be approximately fitted by taking

$$g(J_2', J_3'^2) = J_2'\left(1 - 0.73\frac{J_3'^2}{J_2'^3}\right)$$

in place of $g = J_2'$. This example is by no means occasional. From Theorem 2, we know that it is only a special example in the general case for this kind materials.

References

[1] Jean Martinet, *Singularities of smooth functions and maps*, London Mathematical Society Lecture Note Series 58. Cambridge University Press, 1982.

[2] R. Hill, *The mathematical theory of plasticity*, The Oxford Engineering Series, 1950.

[3] TH. Bröcker, *Differentiable germs and catastrophes*, London Mathematical Society Lecture Note Series 17, Cambridge University Press, 1975.

Whitney 引理的推广及应用

岑燕斌

(黔南师专数学系，贵州都匀 558000)

岑燕明

(贵州民族学院数学系，贵阳 550025)

摘　要

C^∞ 函数芽的局部奇点理论中, Whitney 引理是一个很重要的定理. 本文将证明该定理的整体结论. 基于这一推广, 详细地讨论了一类材料的塑性屈服准则. 发现, 对于这类材料, 塑性屈服准则最一般的形式应是: $g(J_1, J_2', J_3'^2) = 0$. 最后举例加以说明.