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Kronecker Products of Positive Semidefinite Matrices*

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Abstract An $n\times n$ real (not necessarily symmetric) matrix A is positive semidefinite if $xAx^T\geq 0$ for each nonzero n-dimensional real row vector x. A necessary and sufficient condition for the kronecker product of two positive semidefinite (not necessarily symmetric) matrices to be positive semidefinite is given in this paper.

Key words positive semidefinite (not necessarily symmetric) matrix, Kronecker product

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1. Introduction

Let $S^{n\times n}$ denote the set of all $n\times n$ symmetric real matrices, $\mathbf{R}^{n\times n}$ the set of all $n\times n$ real matrices, and $\mathbf{R}^n$ the space of n-dimensional real row vectors. For a given $A\in\mathbf{R}^{n\times n}$, $S(A)=\frac{1}{2}(A+A^T)$ and $K(A)=\frac{1}{2}(A-A^T)$ are the symmetric part and the skew symmetric part of A, respectively, where A^T is the transpose of A. The rank and determinant of A are denoted by $r=r(A)$ and $det(A)$, respectively. The determinant factor with order r of the λ-matrix $\lambda S(A)+K(A)$ is denoted by $D_r(\lambda)$. Furthermore, denote

$$m=m(A)=r(A)-r(K(A)),$$
$$p=p(A)=r(S(A))-\deg(D_r(\lambda)),$$
$$q=q(A)=\frac{1}{2}(r(A)-2r(S(A))+\deg(D_r(\lambda))),$$
$$k=k(A)=\frac{1}{2}(\deg(D_r(\lambda))-r(A)+r(K(A))),$$

where $deg(D_r(\lambda))$ is the degree of the λ-polynomial $D_r(\lambda)$. Recall (see [1]) that A is positive semidefinite (resp. positive definite) if $xAx^T\geq 0$ (resp. $xAx^T>0$) for each non-zero $x\in\mathbf{R}^n$. For $A,B\in\mathbf{R}^{n\times n}$, A and B are congruent if there exists a non-singlar $P\in\mathbf{R}^{n\times n}$ such that $B=PAP^T$.

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In 1970, C. R. Johnson[1] first investigated the positive definite (not necessarily symmetric) matrices. J. S. Li[2] established the canonical forms of the positive definite matrices under congruence. J. S. Li[3] gave further the canonical forms of the positive semidefinite matrices under congruence.

It is well known that the Kronecker product of two positive semidefinite symmetric matrices is still positive semidefinite. The result for positive semidefinite (not necessarily symmetric) matrices is not necessarily true. For example, take $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then, $xAx^T = 0$ for each non-zero $x \in \mathbf{R}^2$. This implies that A is positive semidefinite. It is easy to calculate that $x(A \otimes A)x^T = 2(x_1x_4 - x_2x_3)$ for each $x = (x_1, x_2, x_3, x_4) \in \mathbf{R}^4$. Hence, if $x = (0, 1, 1, 0)$, then $x(A \otimes A)x^T = -2 < 0$. So $A \otimes A$ is not positive semidefinite. The purpose of this paper is to give a necessary and sufficient condition for the Kronecker product of two positive semidefinite (not necessarily symmetric) matrices to be positive semidefinite.

2 Main results

We need the following lemmas.

Lemma 2.1 *Let $A \in \mathbf{R}^{n\times n}$ be positive semidefinite. Then A is congruent to the canonical form:*

$$\tilde{A} = \operatorname{diag}(I_m, A_1, A_2, A_3, O_{n-r}), \tag{1}$$

where $I_m \in \mathbf{R}^{m\times m}$ and $O_{n-r} \in \mathbf{R}^{(n-r)\times(n-r)}$ are the identity matrix and the zero matrix, respectively,

$$A_1 = \operatorname{diag}\left(\begin{bmatrix} 1 & a_1 \\ -a_1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & a_2 \\ -a_2 & 1 \end{bmatrix}, \cdots, \begin{bmatrix} 1 & a_k \\ -a_k & 1 \end{bmatrix}\right) \in \mathbf{R}^{2k\times 2k},$$

$$A_2 = \operatorname{diag}\left(\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}\right) \in \mathbf{R}^{2p\times 2p},$$

$$A_3 = \operatorname{diag}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) \in \mathbf{R}^{2q\times 2q},$$

and $\pm ia_1, \pm ia_2, \cdots, \pm ia_k$ are all of the non-zero roots of $D_r(\lambda)$, $a_1 \geq a_2 \geq \cdots \geq a_k > 0$.

Proof See [3].

Lemma 2.2 *If $A \in \mathbf{R}^{n\times n}$ is positive definite, then $r(A) = r(S(A)) = deg(D_\lambda(\lambda))$, and A is congruent to the block diagonal matrix:*

$$\tilde{A} = \operatorname{diag}(I_m, A_1), \tag{2}$$

where the sense of A_1 is the same as Lemma 2.1.

Proof Since A is positive definite, we have $r = r(A) = n$ and the diagonal blocks of the forms $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ do not occur in the canonical form $\tilde{A}$.

Lemma 2.3 *Let $A \in \mathbf{R}^{n\times n}$ be positive semidefinite. Then the following assertions hold:*

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(1) *All roots of* $D_r(\lambda)$ *are zero if and only if*

$$\deg(D_r(\lambda)) = r(A) - r(K(A));$$

(2) *The block* $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ *does not occur in the canonical form* $\tilde{A}$ *of* A *if and only if*

$$\deg(D_r(\lambda)) = r(S(A));$$

(3) *The block* $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ *does not occur in the canonical form* $\tilde{A}$ *of* A *if and only if*

$$r(A) - r(S(A)) = r(S(A)) - \deg(D_r(\lambda)).$$

Proof This is an immediate corollary of Lemma 2.1.

Lemma 2.4 (1) *Let* $A \in \mathbf{R}^{n\times n}$ *and* $B \in \mathbf{R}^{t\times t}$. *Then* $\mathrm{diag}(A, B)$ *is positive semidefinite if and only if* A *and* B *are positive semidefinite;*

(2) *If* $A, B \in \mathbf{R}^{n\times n}$ *are congruent and* A *is positive semidefinite, then* B *is also positive semidefinite;*

(3) *If* $A, C \in \mathbf{R}^{n\times n}$ *are congruent, and* $B, D \in \mathbf{R}^{t\times t}$ *are congruent, then* $A \otimes B$ *and* $C \otimes D$ *are congruent.*

Proof The proof is easy, therefore is omitted.

Lemma 2.5 *Let* $a > 0$ *and* $b > 0$. *Then*

(1) $\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & b \\ -b & 1 \end{bmatrix}$ *is positive semidefinite (resp. positive definite) if and only if* $ab \leq 1$ (*resp.* $ab < 1$);

(2) *All following matrices are not positive semidefinite:*

$$\begin{gathered}
\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \\
\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & b \\ -b & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \\
\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & b \\ -b & 1 \end{bmatrix}, \\
\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\end{gathered} \tag{3}$$

Proof Denote $A = \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & b \\ -b & 1 \end{bmatrix}$. Then $S(A)$ is permutation similar (denoted by "$\sim$") to the block diagonal matrix:

$$S = \mathrm{diag}\left(\begin{bmatrix} 1 & ab \\ ab & 1 \end{bmatrix}, \begin{bmatrix} 1 & -ab \\ -ab & 1 \end{bmatrix}\right).$$

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S is positive semidefinite (resp. positive definite) if and only if $\det S = 1 - a^2b^2 \geq 0$ (resp. $1 - a^2b^2 > 0$). This implies that (1) holds.

The proof of (2) is similar.

Theorem 2.6 *Let $A \in \mathbf{R}^{n\times n}$ and $B \in S^{t\times t}$ be positive semidefinite. Then $A \otimes B$ is positive semidefinite.*

Proof From Lemma 2.1, A is congruent to the cannonical form $\tilde{A}$. Moreover, B is congruent to the diagonal matrix $\tilde{B} = \operatorname{diag}(I_s, O_{t-s})$, where $s = r(B)$. It follows from Lemma 2.4 that $A \otimes B$ is congruent to the matrix:

$$\tilde{A} \otimes \tilde{B} = \operatorname{diag}(I_m \otimes \tilde{B}, A_1 \otimes \tilde{B}, A_2 \otimes \tilde{B}, A_3 \otimes \tilde{B}, O_{n-l}\tilde{B}).$$

Since

$$\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \otimes \tilde{B} \sim \operatorname{diag}\left(\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}, \cdots, \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}, O_{2t-2s}\right) \in \mathbf{R}^{2t\times 2t},$$

$\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \otimes \tilde{B}$ is positive semidefinite. From Lemma 2.4, $A_1 \otimes \tilde{B}$ is positive semidefinite. Similarly,

$$\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \otimes \tilde{B} \sim \operatorname{diag}\left(\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, O_{2t-2s}\right) \in \mathbf{R}^{2t\times 2t}.$$

Hence, from Lemma 2.4, $A_2 \otimes \tilde{B}$ is positive semidefinite. In addition,

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \tilde{B} \sim \operatorname{diag}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, O_{2t-2s}\right) \in \mathbf{R}^{2t\times 2t}.$$

Consequently, $A_3 \otimes \tilde{B}$ is positive semidefinite. Finally, $I_m \otimes \tilde{B} \sim \operatorname{diag}(I_{ms}, O_{m(t-s)})$. So $I_m \otimes \tilde{B}$ is positive semidefinite, too. From Lemma 2.4, $\tilde{A} \otimes \tilde{B}$ is positive semidefinite. Thus $A \otimes B$ is positive semidefinite.

Corollary 2.7 *Let $A \in \mathbf{R}^{n\times n}$ and $B \in S^{t\times t}$ be positive definite. Then $A \otimes B$ is positive definite.*

Proof From Lemma 2.2, A is congruent to the block diagonal matrix:

$$\tilde{A} = \operatorname{diag}(I_m, A_1).$$

Moreover, B is congruent to I_t. From Lemma 2.4, $A \otimes B$ is congruent to the matrix:

$$\tilde{A} \otimes I_t = \operatorname{diag}(I_m \otimes I_t, A_1 \otimes I_t).$$

Clearly, $I_m \otimes I_t = I_{mt}$ and $A_1 \otimes I_t$ are positive definite. Thus $\tilde{A} \otimes I_t$ is positive definite.

Theorem 2.8 *Let $A \in S^{n\times n}$ and $B \in \mathbf{R}^{t\times t}$ be positive semidefinite. Then $A \otimes B$ is positive*

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semidefinite.

Proof The proof is similar to that of Theorem 2.6 and is omitted.

Theorem 2.9 *Let* $A \in \mathbf{R}^{n\times n}$ *and* $B \in \mathbf{R}^{t\times t}$ *be positive semidefinite and non-symmetric. Then* $A \otimes B$ *is positive semidefinite if and only if the following conditions hod:*

(1) $r(A) = r(S(A)) = deg(D_r(\lambda)) = r$, *and* $r(B) = r(S(B)) = deg(D_s(\lambda)) = s$, *where* $D_r(\lambda)$ *and* $D_s(\lambda)$ *are the determinant factors with order r and order s of the* λ*-matrices* $\lambda S(A) + K(A)$ *and* $\lambda S(B) + K(B)$, *respectively*;

(2) *If* α *and* β *are the roots of* $D_r(\lambda)$ *and* $D_s(\lambda)$, *respectively, then* $|\alpha\beta| \le 1$.

Proof We first prove that $A \otimes B$ is congruent to a block diagonal matrix G whose diagonal blocks are of the following forms:

$$I_f, \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & b \\ -b & 1 \end{bmatrix}, \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & b \\ -b & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & b \\ -b & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and the zero matrix O_g, where $a > 0$ and $b > 0$.

Since A is congruent to the canonical form $\tilde{A}$ and B is congruent to the following canonical form:

$$\tilde{B} = \operatorname{diag}(I_{m'}, B_1, B_2, B_3, O_{(t-s)}),$$

where

$$B_1 = \operatorname{diag}\left(\begin{bmatrix} 1 & b_1 \\ -b_1 & 1 \end{bmatrix}, \cdots, \begin{bmatrix} 1 & b_{k'} \\ -b_{k'} & 1 \end{bmatrix}\right) \in \mathbf{R}^{2k'\times 2k'},$$

$$B_2 = \operatorname{diag}\left(\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}\right) \in \mathbf{R}^{2p'\times 2p'},$$

$$B_3 = \operatorname{diag}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) \in \mathbf{R}^{2q'\times 2q'}.$$

$A \otimes B$ is congruent to the matrix:

$$\tilde{A} \otimes \tilde{B} = \operatorname{diag}(I_m \otimes \tilde{B}, A_1 \otimes \tilde{B}, A_2 \otimes \tilde{B}, A_3 \otimes \tilde{B}, O_{n-r} \otimes \tilde{B}).$$

Clearly, we have

$$\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \otimes \tilde{B} = \begin{bmatrix} \tilde{B} & a\tilde{B} \\ -a\tilde{B} & \tilde{B} \end{bmatrix} \sim$$

$$\operatorname{diag}\left(\begin{bmatrix} I_{m'} & aI_{m'} \\ -aI_{m'} & I_{m'} \end{bmatrix}, \begin{bmatrix} B_1 & aB_1 \\ -aB_1 & B_1 \end{bmatrix}, \begin{bmatrix} B_2 & aB_2 \\ -aB_2 & B_2 \end{bmatrix}, \begin{bmatrix} B_3 & aB_3 \\ -aB_3 & B_3 \end{bmatrix}, O_{2(t-s)}\right).$$

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Note that

$$\begin{bmatrix} I_{m'} & aI_{m'} \\ -aI_{m'} & I_{m'} \end{bmatrix} \sim \operatorname{diag}\left(\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}, \cdots, \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \right) \in \mathbf{R}^{2m' \times 2m'},$$

$$\begin{bmatrix} B_1 & aB_1 \\ -aB_1 & B_1 \end{bmatrix} \sim \operatorname{diag}\left(\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & b_1 \\ -b_1 & 1 \end{bmatrix}, \cdots, \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & b_{k'} \\ -b_{k'} & 1 \end{bmatrix} \right),$$

$$\begin{bmatrix} B_2 & aB_2 \\ -aB_2 & B_2 \end{bmatrix} \sim \operatorname{diag}\left(\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \right),$$

and

$$\begin{bmatrix} B_3 & aB_3 \\ -aB_3 & B_3 \end{bmatrix} \sim \operatorname{diag}\left(\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right).$$

Hence $\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \otimes \widetilde{B}$ is permutation similar to such a block diagonal matrix that its diagonal blocks are of the forms:

$$\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}, \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & b \\ -b & 1 \end{bmatrix}, \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \text{ and the zero block } O.$$

Similarly, $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \otimes \widetilde{B}$ is permutation similar to such a block diagonal matrix that its diagonal blocks are of the forms:

$$\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & b \\ -b & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \text{ and the zero block } O.$$

And $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \otimes \widetilde{B}$ is permutation similar to a block diagonal matrix whose diagonal blocks are of the following forms:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & b \\ -b & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \text{ and the zero block } O.$$

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Finally, $I_m \otimes \tilde{B} = \operatorname{diag}(\tilde{B}, \cdots, \tilde{B}) \in \mathbf{R}^{mt \times mt}$ is permutation similar to a block diagonal matrix whose diagonal blocks are of the forms:

$$I, \begin{bmatrix} 1 & b \\ -b & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } O.$$

Thus the assertion holds.

Now suppose that $A \otimes B$ is positive semidefinite. By the proof above, $A \otimes B$ is congruent to a block diagonal matrix G. From Lemma 2.4, G is positive semidefinite. It follows from Lemma 2.5 that G does not contain the diagonal blocks having one of the forms in (3). If $\tilde{A}$ contains the diagonal block $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$, then $\tilde{B}$ does not contain the blocks having the forms $\begin{bmatrix} 1 & b \\ -b & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. In other words, $\tilde{B}$ is symmetric. Hence, B is symmetric, a contradiction. This implies that $\tilde{A}$ does not contain the block A_2. Similarly, $\tilde{A}$ does not contain the block A_3. Therefore,

$$\tilde{A} = \operatorname{diag}(I_m, A_1, O_{n-r}). \tag{i}$$

By Lemma 2.3, we have $r = r(A) = r(S(A)) = deg(D_A(\lambda))$. Similary, we have

$$\tilde{B} = \operatorname{diag}(I_{m'}, B_1, O_{n-s}). \tag{ii}$$

and $s = r(B) = r(S(B)) = deg(D_s(\lambda))$. This implies that (1) holds. In addition, similar to the proof above, $A \otimes B$ is congruent to the block diagonal matrix whose diagonal blocks are of the following forms:

$$I, \begin{bmatrix} 1 & a_f \\ -a_f & 1 \end{bmatrix}, \begin{bmatrix} 1 & b_g \\ -b_g & 1 \end{bmatrix}, \begin{bmatrix} 1 & a_f \\ -a_f & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & b_g \\ -b_g & 1 \end{bmatrix},\ 1 \le f \le k, 1 \le g \le k', \tag{iii}$$

and the zero matrix O. Therefore, $\begin{bmatrix} 1 & a_f \\ -a_f & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & b_g \\ -b_g & 1 \end{bmatrix}$ is positive semidefinite. From Lemma 2.5, $a_f b_g \le 1$. Since

$$D_A(\lambda) = det(\lambda S(\tilde{A}) + K(\tilde{A})) = \lambda^m (\lambda^2 + a_1^2) \cdots (\lambda^2 + a_k^2),$$

the non-zero roots of $D_r(\lambda)$ are $\pm ia_1, \pm ia_2, \cdots, \pm ia_k$. Similarly, the non-zero roots of $D_s(\lambda)$ are $\pm ib_1, \pm ib_2, \cdots, \pm ib_{k'}$. Consequently, (2) holds.

Finally, suppose (1) and (2) are satisfied. It follows from (1) and Lemma 2.3, A and B are congruent to the matrices (i) and (ii), respectively. Hence $A \otimes B$ is congruent to

$$\tilde{A} \otimes \tilde{B} = \operatorname{diag}(I_m \otimes \tilde{B}, A_1 \otimes \tilde{B}, O_{n-r} \otimes \tilde{B}).$$

Hence we obtain that $\tilde{A} \otimes \tilde{B}$ is congruent to a block diagonal matrix G whose diagonal blocks are of the forms (iii). It follows from (2) and Lemma 2.4 that G is positive semidefinite. This implies that $A \otimes B$ is positive semidefinite.

This completes the proof.

Note that if $A \in \mathbf{R}^{n \times n}$ is positive definite, then $D_r(\lambda) = det(\lambda S(A) + K(A))$. Hence, we

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have

Theorem 2.10 *Let $A\in\mathbf{R}^{n\times n}$ and $B\in\mathbf{R}^{t\times t}$ be positive definite. Then $A\otimes B$ is positive definite if and only if $|\alpha\beta|<1$ for each root α of $\det(\lambda S(A))+K(A))$ and each root β of $\det(\lambda(S(A))+K(A))$.*

Proof The proof is omitted.

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半正定未必对称矩阵的 Kronecker 乘积

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摘　要

一个 $n\times n$ 实矩阵 A 称为半正定, 如果对每个 n 维非零实向量 x, 均有 $xAx^T\geq 0$. 本文给出了两个半正定, 未必对称实矩阵为半正定的充要条件.

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