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An Adjoint Matrix of a Real Idempotent Matrix*

Jin Bai Kim
(Dept. of Math., West Virginia University Morgantown, WV 26506, USA)
Hee Sik Kim
(Dept. of Math. Education Chungbuk National University, Chongju 360-763, Korea)
Seung Dong Kim
(Dept. of Math., Kong-Ju National Teachers' University, Kongju 314-701, Korea)

Abstract We prove that an adjoint matrix of a real idempotent matrix is idempotent.

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1. Introduction

There are papers on real idempotent matrices (for instance [1], [3] and [4]). The first author of this paper in [5] proved that the second ajoint matrix of a Fuzzy idempotent matrix is idempotent. Our motivation of this paper is initiated from [5] and we prove that an adjoint matrix of a real idempotent matrix is idempotent.

2. Lemmas

We have two lemmas in this section. We need some definitions.

Definition 1 (*i*) *R denotes the set of all real numbers. $M_n(R)$ denotes the set of all n by n real matrices.*

(*ii*) *Let $A \in M_n(R)$. A^t denotes the transpose of A.*

(*iii*) *Let $A = (a_{ij}) \in M_n(R)$. The cofactor A_{ij} of a_{ij} in $\det(A)$ is $(-1)^{i+j}$ times the determinant of the submatrix of order $n-1$ obtained by deleting the ith row and the jth column from A, where $\det(A)$ denotes the determinant of A* (see [6, p. 57]).

(*iv*) *If A is a matrix of order n and A_{ij} is the cofactor of a_{ij} in $\det(A)$, then the matrix*

$$\operatorname{adj}(A) = (A_{ij})^t = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

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is called the adjoint matrix of A (*see* [6, *p.* 58] *for adj* (A)).

(*v*) E_{ij} *in* $M_n(R)$ *denotes the matrix obtained from the identity matrix* $I_n = I$ *by interchanging row* i *and row* j$(i \neq j)$.

Lemma 1 *Let* $A \in M_n(R)$ $(n \geq 3)$. *Then we have that*

$$\mathrm{adj}(E_{ij} A E_{ij}) = E_{ij}(\mathrm{adj}(A))E_{ij}.$$

We omit the proof of Lemma 1.

Lemma 2 *We assume that* $n \geq 3$. *Let* $A = (a_{ij}) \in M_n(R)$ *be a real matrix of order n defined by*

$$a_{ij} = \begin{cases} 1 & \text{if } 1 \leq i = j \leq n - 2, \\ 0 & \text{if } i \neq j \text{ and } 1 \leq i, j \leq n - 2, \\ 0 & \text{if } n - 1 \leq i, j \leq n. \end{cases}$$

Assume that A *is idempotent. Then we have the following*:

(*i*) *If* $a_{n-1,i} \neq 0$ *for* $1 \leq i \leq n-2$, *then* $a_{j,n-1} = 0$ *for* $1 \leq j \leq n-3$, *and* $a_{in} = 0$.

(*ii*) *If* $a_{ni} \neq 0$ *for* $1 \leq i \leq n-2$, *then* $a_{jn} = 0$ *for* $1 \leq j \leq n-3$, *and* $a_{i,n-1} = 0$.

(*iii*) *If* $a_{i,n-1} \neq 0$ *for* $1 \leq i \leq n-1$, *then* $a_{n-1,j} = 0$ *for* $1 \leq j \leq n-3$ *and* $a_{ni} = 0$.

(*iv*) *If* $a_{1n} \neq 0$ *for* $1 \leq i \leq n-2$, *then* $a_{nj} = 0$ *for* $1 \leq j \leq n-3$, *and* $a_{n-1,i} = 0$.

We omit the proof of Lemma 2.

Example 1 We list 18 real 5×5 idempotent matrices A with $r(A) = 3$ (the rank of A is equal to 3). In this example, $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & z & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a & b & 0 & 0 & 0 \end{bmatrix}$ denotes a 5×5 real idempotent matrix A with $r(A) = 5$ and $a \neq 0$, $b \neq 0$, and $z \neq 0$. In this matrix, we can add that:

(i) If $a \neq 0$, $b \neq q$ 0 and all other entries of A_4 and A_5 are zero, then we can have $a_{34} = z \neq 0$ and all other entries of ${}_4A$ and ${}_5A$ must be zero, where A_i and ${}_iA$ denote respectively the ith row and ith column of A.

(ii) Similarly, if $z \neq 0$ and all other entries of ${}_4A$ and ${}_5A$ are zero, then only non-zero entries are $a = a_{51}$ and $b = a_{52}$.

$$\overset{(1)}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & v \\ 0 & 0 & 1 & 0 & w \\ a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \quad \overset{(2)}{\begin{bmatrix} 1 & 0 & 0 & 0 & u \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & w \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \quad \overset{(3)}{\begin{bmatrix} 1 & 0 & 0 & 0 & u \\ 0 & 1 & 0 & 0 & v \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

$$\text{(4)}\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & w \\ a & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(5)}\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & v \\ 0 & 0 & 1 & 0 & 0 \\ a & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(6)}\ \begin{bmatrix} 1 & 0 & 0 & 0 & u \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & b & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{(7)}\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ a & b & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(8)}\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a & b & c & 0 & 0 \end{bmatrix} \quad \text{(9)}\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & y & 0 \\ 0 & 0 & 1 & z & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{(10)}\ \begin{bmatrix} 1 & 0 & 0 & x & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & z & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \end{bmatrix} \quad \text{(11)}\ \begin{bmatrix} 1 & 0 & 0 & x & 0 \\ 0 & 1 & 0 & y & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 \end{bmatrix} \quad \text{(12)}\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & z & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a & b & 0 & 0 & 0 \end{bmatrix}$$

$$\text{(13)}\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & y & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a & 0 & c & 0 & 0 \end{bmatrix} \quad \text{(14)}\ \begin{bmatrix} 1 & 0 & 0 & x & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & b & c & 0 & 0 \end{bmatrix} \quad \text{(15)}\ \begin{bmatrix} 1 & 0 & 0 & x & 0 \\ 0 & 1 & 0 & y & 0 \\ 0 & 0 & 1 & z & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{(16)}\ \begin{bmatrix} 1 & 0 & 0 & 0 & x \\ 0 & 1 & 0 & 0 & y \\ 0 & 0 & 1 & 0 & z \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(17)}\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ a & b & c & 0 & 0 \\ d & e & f & 0 & 0 \end{bmatrix} \quad \text{(18)}\ \begin{bmatrix} 1 & 0 & 0 & x & u \\ 0 & 1 & 0 & y & v \\ 0 & 0 & 1 & z & w \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3. Theorem

We quote the following [2]:

If A is a real idempotent matrix of order n, then A is similar to a diagonal matrix diag $(1, 1, \cdots, 1, 0, \cdots, 0)$, that is $A = T(\mathrm{diag}(1, 1, \cdots, 1, 0, \cdots, 0))T^{-1}$, where T is a non-singular matrix of order n.

We prove the following theorem.

Theorem *Let $A = (a_{ij}) \in M_n(R)$ be a real idempotent matrix. Then the adjoint matrix $\mathrm{adj}(A)$ is idempotent.*

Proof The proof consists of several steps

(i) Suppose the rank $r(A)$ of a real idempotent matrix A of order n is equal to n. Then we see that $A = I_n = I$, the identity matrix. We can compute $\mathrm{adj}(A)$ as I and hence $\mathrm{adj}(A) = I$ is idempotent.

(ii) Suppose that $r(A) = n-1$. Then we can assume, without loss of generality, that

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ a_1 & a_2 & a_3 & \cdots & a_{n-1} & 0 \end{bmatrix}$$

We can compute adj(A), the adjoint matrix of A as follows:

$$\mathrm{adj}(A) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} & 0 \end{bmatrix}$$

We can show that adj(A) is idempotent.

(iii) We referring to Lemma 1 and 2, and Example 1, we claim that if A is an idempotent of rank $(n-2)$, then one of the following three statements holds:

(1) A has two rows (A_{n-1} and A_n) each of two is the zero vector.

(2) A has two columns (${}_{n-1}A$ and ${}_nA$) each of two is the zero vector.

(3) A has one row and one column (A_{n-1} and ${}_nA$, or A_n and ${}_{n-1}A$) both of them are zero vectors.

(iv) We just prove that the claim mentioned in the above (iii) is true (for a case). Suppose that A is an idempotent of rank of $(n-2)$ and suppose, in addition, that $a_{ii} = 1$ for $i = 1, 2, \cdots, n-2$ and $a_{ii} = 0$ for $i = n-1$ and $i = n$. Then we can show that $a_{ij} = 0$ for $i \neq j$ and $1 \leq i, j \leq n-3$, and $a_{ij} = 0$ for $i \neq j$ and $n-1 \leq i, j \leq n$. Now if $a_{n-1,i} \neq 0 (1 \leq i \leq n-2)$, then ${}_{n-1}A$, the $n-1$ column, must be the zero vector. In addition, if $a_{nj} \neq 0 (1 \leq j \leq n-2)$ then ${}_nA$ (the n column) must also be the vector. This proves the claim for a case (referring to 17, Example 1).

The rest of all other cases will be proved by a similar way using Lemma 2.

Now we see that adj$(A) = 0 \in M_n(R)$ for an idempotent matrix of order $(n-2)$, where 0 denotes the zero matrix. Therefore adj(A) is idempotent.

(v) Let A be an idempotent matrix of rank k, where $k \leq n-3$. We again refer to Lemmas 1 and 2, and Example 1, and we easily deduce that adj$(A) = 0$ when A is an idempotent of rank $k (k \leq n-3)$. We know that adj$(A) = 0$ is idempotent. This proves Theorem.

References

[1] J. A. Erdos, *On product of idempotent matrices*, Glasgow Math. Journal (1967), 118-122.

[2] F. R. Gantmacher, *The theory of matrices*, Volume 1, (1959), Example 2, p. 226; Chelsea Publishing Company, New York.

[3] J. H. Hodges, *Idempotent matrices*, Amer. Math. Monthly, 1966, 277.

[4] Jin Bai Kim, *Idempotent generated Rees matrix semigroups*, Kyungpook Mathematical, Journal **10**: **1** (1970), 7-14.

[5] Jin Bai Kim and K. H. Choi, *The second adjoint matrix of a Fuzzy idempotent matrix*, The Journal of Fuzzy Mathematics, **2**: **2** (1994), 341-350.

[6] C. C. Macduffee, *Vectors and matrices*, The Mathematical Association of America, 1943.