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On n-Widths for Some CVD Matrices*

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Abstract In this paper we prove that $d_{2k}=\delta_{2k}=d^{2k}\geq b_{2k}$, where d_{2k}, δ_{2k}, b_{2k} denote the Kolmogorov, linear, Bernstein $2k$-widths of $A\ (Bl_p^M)$ in l_q^N, d^{2k} denotes the Gelfand $2k$-width of $A^T(Bl_{q'}^N)$ in $l_{p'}^M$, respectively. Bl_p^M denotes the unit ball of l_p^M. A is a $N\times M$ CVD matrix ($N>M=\operatorname{rank}A$, M is odd). $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{q}+\frac{1}{q'}=1$ $(1\leq q\leq p<+\infty, p\neq 1)$.

Key words CVD matrix, (p,q) spectrum couples, $2k$-width, sign-consistent.

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1. Introduction

Let A be an $N\times M$ matrix, abbreviated as $A\in\mathbf{M}^{N\times M}$ with column vectors $a^1,\cdots,a^M$. For $1\leq p,q<+\infty$, set

$$l_p^M:=\{x\in\mathbf{R}^M:\|x\|_p=(\sum_{i=1}^M|x_i|^p)^{\frac{1}{p}}<+\infty\},$$

$$l_q^N:=\{x\in\mathbf{R}^N:\|x\|_q=(\sum_{i=1}^N|x_i|^q)^{\frac{1}{q}}<+\infty\},$$

$$Bl_p^M:=\{x\in l_p^M:\|x\|_p\leq 1\}.$$

The n-width, in the sense of Kolmogorov, of $A\ (Bl_p^M)$ in l_q^N is given by

$$d_n(A,l_p^M,l_q^N):=\inf_{L_n}\sup_{\|x\|_p\leq 1}\inf_{y\in L_n}\|Ax-y\|_q,$$

where L_n is any subspace of l_q^N of dimension n.

The linear n-width of $A\ (Bl_p^M)$ in l_q^N is given by

$$\delta_n(A,l_p^M,l_q^N):=\inf_{P_n}\sup_{\|x\|_p\leq 1}\|Ax-P_nx\|_q,$$

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where P_n runs over all $N\times M$ matrices of rank n.

The n-width of $A(B l_p^M)$ with respect to l_q^N, in the sense of Gelfand, is defined by

$$d^n(A, l_p^M, l_q^N) := \inf_{X_n} \sup_{\substack{\|x\|_p\le 1 \\ x\perp X_n}} \|Ax\|_q,$$

where X_n is any subspace of l_p^M of dimension n.

The n-width of $A(B l_p^M)$ in l_p^N, in the sense of Bernstein, is given by

$$b_n(A, l_p^M, l_q^N) := \sup_{X_{n+1}} \inf_{\substack{x\in X_{n+1} \\ \|x\|_p\le 1}} \|Ax\|_q,$$

where X_{n+1} is any subspace of l_p^M of dimension $n+1$.

For $A\in STP_l(N\times M)$, $l=\min(N, M)$, and $1\le q\le p<+\infty, p\ne 1$, Buslaev[1] proved that

$$d_n(A, l_p^M, l_q^N) = \delta_n(A, l_p^M, l_q^N) = d^n(A^T, l_{q'}^N, l_{p'}^M) \ge b_n(A, l_p^M, l_q^N).$$

In this paper, for $A\in CVD(N\times M)$ ($N>M=\operatorname{rank}A$, M is odd), and $1\le q\le p<+\infty$, $p\ne 1$, we present the corresponding results (for $2k$-widths).

2 Preliminaries

Definition 1[2] *Let $A=(a_{ij})\in\mathbf{M}^{N\times M}$. If all nozero $k\times k$ minors of A have the same sign, A is said to be sign-consistent of order k, abbreviated by $A\in SC_k(N\times M)$. If all $k\times k$ minors of A are nozero and have the same sign, A is said to be strictly sign-consistent of order k, abbreviated by $A\in SSC_k(N\times M)$.*

Let $x=(x_1, x_2, \cdots, x_M)\in\mathbf{R}^M$. Denote by $S^-(x)$ the number of sign changes in theseqence obtained from x_1, x_2, $\cdots$, x_M by deleting all zero terms, by $S^+(x)$ the maximum numbers of sign changes possible in the vector x by allowing each zero to be replaced by $+1$ or -1. We define that

$$S_c^+(x) := \max_K S^+(x_K, x_{K+1}, \cdots, x_M, x_1, \cdots, x_K),$$

$$S_c^-(x) := \max_K S^-(x_K, x_{K+1}, \cdots, x_M, x_1, \cdots, x_K).$$

Definition 2[2] *Let $A\in\mathbf{M}^{N\times M}$. If for any $y\in\mathbf{R}^M$, $S_c^-(Ay)\le S_c^-(y)$, then A is said to be cyclic variation-diminishing, abbreviated by $A\in CVD(N\times M)$.*

Lemma 1[2] *Let $A\in\mathbf{M}^{N\times M}$, $M=2k-1, N>M=\operatorname{rank}A$. Then*

(*i*) $A\in CVD(N\times M)\Leftrightarrow A\in SC_{2r-1}$, $r=1, 2, \cdots, k$;

(*ii*) $A\in SSC_{2r-1}\Leftrightarrow S_c^+(Ay)\le S_c^-(y)$, $\forall y\ne 0$.

Lemma 2[2] *If $A\in SC_k(N\times M)$, $0<k\le\min(N, M)$, $\operatorname{rank}A\ge k$, then there exists*

$\{A_\sigma\}_{\sigma\geq 0}\subset SSC_k$ *such that* $A_\sigma\to A$ $(\sigma\to+\infty)$.

Lemma 3 *Let* $A\in \mathbf{M}^{N\times M}$. *If* $\{A_\sigma\}_{\sigma=1}^{+\infty}\subset \mathbf{M}^{N\times M}$ *and* $A_\sigma\to A$ $(\sigma\to+\infty)$, *then*

$$d_n(A_\sigma, l_p^M, l_q^N)\to d_n(A, l_p^M, l_q^N)\ (\sigma\to+\infty),$$
$$\delta_n(A_\sigma, l_p^M, l_q^N)\to \delta_n(A, l_p^M, l_q^N)\ (\sigma\to+\infty),$$
$$d^n(A_\sigma, l_p^M, l_q^N)\to d^n(A, l_p^M, l_q^N)\ (\sigma\to+\infty),$$
$$b_n(A_\sigma, l_p^M, l_q^N)\to b_n(A, l_p^M, l_q^N)\ (\sigma\to+\infty).$$

Proof We only prove that $d_n(A_\sigma, l_p^M, l_q^N)\to d_n(A, l_p^M, l_q^N)$ $(\sigma\to+\infty)$, the proof for the others are similar. By the definition and the duality theorem of the best approximation, we have

$$d_n(A_\sigma, l_p^M, l_q^N) := \inf_{L_n}\sup_{\|x\|_p\leq 1}\inf_{y\in L_n}\|A_\sigma x - y\|_q$$
$$= \inf_{L_n}\sup_{\|x\|_p\leq 1}\sup_{\substack{f\perp L_n\\ \|f\|_{q'}=1}}|(f, A_\sigma x)|\ \left(\frac{1}{q}+\frac{1}{q'}=1\right)$$

and

$$d_n(A, l_p^M, l_q^N) = \inf_{L_n}\sup_{\|x\|_p\leq 1}\sup_{\substack{f\perp L_n\\ \|f\|_{q'}=1}}|(f, Ax)|.$$

We only need to prove that

$$\sup_{\|x\|_p\leq 1}\sup_{\substack{f\perp L_n\\ \|f\|_{q'}=1}}|(f, A_\sigma x)|\to \sup_{\|x\|_p\leq 1}\sup_{\substack{f\perp L_n\\ \|f\|_{q'}=1}}|(f, Ax)|\ (\sigma\to+\infty)$$

holds uniformly for L_n.

Since

$$|(f, A_\sigma x)|\leq |(f, (A_\sigma - A)x)| + |(f, Ax)|\leq \|f\|_{q'}\|(A_\sigma - A)x\|_q + |(f, Ax)|$$
$$|(f, Ax)|\leq \|f\|_{q'}\|(A_\sigma - A)x\|_q + |(f, Ax)|,$$

so $\forall x\in Bl_p^M$, we have

$$\sup_{\substack{f\perp L_n\\ \|f\|_{q'}=1}}|(f, A_\sigma x)|\leq \|(A_\sigma - A)x\|_q + \sup_{\substack{f\perp L_n\\ \|f\|_{q'}=1}}|(f, Ax)|,$$

$$\sup_{\substack{f\perp L_n\\ \|f\|_{q'}=1}}|(f, Ax)|\leq \|(A_\sigma - A)x\|_q + \sup_{\substack{f\perp L_n\\ \|f\|_{q'}=1}}|(f, A_\sigma x)|,$$

thus

$$\left|\sup_{\|x\|_p\leq 1}\sup_{\substack{f\perp L_n\\ \|f\|_{q'}=1}}|(f, A_\sigma x)| - \sup_{\|x\|_p\leq 1}\sup_{\substack{f\perp L_n\\ \|f\|_{q'}=1}}|(f, Ax)|\right|$$
$$\leq \sup_{\|x\|_p\leq 1}\|(A_\sigma - A)x\|_q \to 0\ (\sigma\to+\infty).$$

Therefore

$$\sup_{\|x\|_p\le 1}\sup_{\substack{f\perp L_n\\ \|f\|_{q'}=1}}|(f,A_\sigma x)|\to\sup_{\|x\|_p\le 1}\sup_{\substack{f\perp L_n\\ \|f\|_{q'}=1}}|(f,A_x)|\,(\sigma\to+\infty)$$

holds unfomly for L_n.

So

$$d_n(A_\sigma, l_p^M, l_q^N)\to d_n(A, L_p^M, l_q^N)\quad(\sigma\to+\infty).$$

3 The main results

Definition 3[1] *Let* $A\in\mathbf{M}^{N\times M}$, $z\in\mathbf{R}^M$, $\lambda\in\mathbf{R}$, $1\le p, q<+\infty$. *If*

$$A^T(Az)_{(q)}=\lambda^q(z)_{(p)},\quad \|z\|_p=1,$$

then we say that (z,λ) *is a* (p,q) *spectrum couples of* A, *denoted by* $(z,\lambda)\in SP(A,p,q)$. *where* $(a)_{(s)}=|a|^{s-1}\mathrm{sgn}\,a$ *for real numbers a and s*,

$$(z)_{(s)}=(|z_1|^{s-1}\mathrm{sgn}z_1, |z_2|^{s-1}\mathrm{sgn}z_2,\cdots, |z_M|^{s-1}\mathrm{sgn}z_M)$$

for vector $z=(z_1, z_2,\cdots, z_M)$. z *is called a spectrum vector of* A, λ *is a spectrum number of* A.

Theorem 1 *For* $A\in\mathbf{M}^{N\times M}$, M *is odd*, $\mathrm{rank}\,A=M$, *if* $A\in SSC_{2r-1}$, $r=1,2,\cdots,\frac{M+1}{2}$, $\kappa\in[0,\frac{M+1}{2})\cap\mathbf{Z}$, $1<p,q<+\infty$, *then*

$$\mathrm{SP}_{2k}(A,p,q)_c:=\{(z,\lambda)\in\mathrm{SP}(A,p,q):S_c(z)=k\}\ne\varnothing,$$

where $S_c(z)$ *denotes* $S_c^+(z)=S_c^-(z)$.

Proof For $k=0$, the proof is easy (see [1]). We only prove the case of $k\ge 1$.

Let

$$O^{2k}:=\{y\in\mathbf{R}^{2k+1}:y=(y_1,\cdots,y_{2k+1}),\ \sum_{i=1}^{2k+1}|y_i|=1\}.$$

For any $y\in O^{2k}$, set

$$\begin{aligned}u(t,y):=\{&\mathrm{sgn}y_1, 0\le t<|y_1|;\cdots,\mathrm{sgn}y_{2k+1},\\ &|y_1|+|y_2|+\cdots+|y_{2k}|\le t\le|y_1|+|y_2|+\cdots+|y_{2k+1}|\},\\ z(y):=&(z_1(y),\cdots,z_M(y)),\end{aligned}$$

where $z_i(y)=M\int_{\frac{i-1}{M}}^{\frac{i}{M}}u(t,y)dt$, $i=1,2,\cdots,M$. We easily see that $S_c^-(z(y))\le 2k$.

Define

$$Z_0(k):=\{z(y):y\in O^{2k}\},\ Z_s(k):=\{z=z_s(y):y\in O^{2k}\}\subset l_p^M,$$

where $z_s(y)$ satisfies $A^T(Az_s(y))_{(q)} = \mu_{s+1}^q(y)(z_{s+1}(y))_{(p)}$ with $\mu_{s+1}(y)$ such that $\|z_s(y)\|_p = 1$, $z_0(y) = z(y)$.

It is easily known that $Z_s(k)$ is an odd, continuous correspondence of O^{2k}. Thus, $\forall s \in \mathbf{Z}^+$, from Borsuk Theorem [1], there exists $\hat{z} \in Z_s(k)$ such that $S_c^+(\hat{z}) \geq 2k$. Set

$$Y_s(k) := \{y \in \mathbf{O}^{2k} : S_c^+(z_s(y)) \geq 2k\} \neq \varnothing.$$

Then $Y_s(k)$ is a closed subset of O^{2k}. Furthermore, $Y_s(k)$ is a compact subset. Obviously, $Y_0(k) \supset Y_1(k) \supset \cdots \supset Y_s(k) \supset \cdots$, so that there exists $y_0 \in \bigcap_{s=0}^{\infty} Y_s(k)$. By similar proof of the [1, Lemma 1], there exists subseqence $\{z_{s_i}(y_0)\}_{s_i \in z^+} \subset \{z_s(y_0)\}_{s \in z^+}$ such that $z_{s_i}(y_0)$ converges to a spectrum vector x of A, μ_{s_i} converges to the corresponce spectrum number λ and

$$\forall s \in \mathbf{Z}^+,\ S_c^+(z_s(y_0)) \geq 2k \Rightarrow S_c^+(x) \geq 2k.$$

By Lemma 1, for any $y \in O^{2k}$, we also have

$$\begin{aligned} S_c^+(z_s(y)) &= S_c^+(A^T(Az_{s-1}(y))_{(q)} \leq S_c^-(Az_{s-1}(y)) \\ &\leq S_c^+(Az_{s-1}(y)) \leq S_c^-(z_{s-1}(y)) \leq \cdots \\ &\leq S_c^-(z_0(y)) = S_c^-(z(y)) \leq 2k. \end{aligned}$$

So

$$S_c^+(z_{s+1}(y_0)) \leq S_c^-(z_s(y_0)) \leq 2k \Rightarrow S_c^+(x) \leq S_c^-(x) \leq 2k.$$

Therefore $S_c^-(x) = S_c^+(x) = S_c(x) = 2k$, i.e. $(x, \lambda) \in \mathrm{SP}_{2k}(A, p, q)_c$. This proves Theorem 1. □

Let

$$\lambda_{2k} := \inf\{\lambda : (z, \lambda) \in \mathrm{SP}_{2k}(A, p, q)_c\}.$$

Theorem 2 *If $p \geq q$, then under the same hypothesis in Theorem 1*

$$d_{2k}(A, l_p^M, l_q^N) = \delta_{2k}(A, l_p^M, l_q^N) = \lambda_{2k} = d^{2k}(A^T, l_{q'}^N, l_{p'}^M) \geq b_{2k}(A, l_p^M, l_q^N)$$

and the characterics of the optimal subspace for d_{2k} (with the optimal matrix for δ_{2k}) are given.

By the similar proof of [1, Theorem 2], we may easily prove that $d_{2k} \geq \lambda_{2k}$. For proving that $\delta_{2k} \leq \lambda_{2k}$, we need the following lemma.

Lemma 4 *Under the assumption of Theorem 2, we have that for any $\forall (z, \lambda) \in SP_{2k}(A, p, q)_c$, there exists $L_{2k} \subset l_q^N$, $\dim L_{2k} \leq 2k$ such that*

$$d(A(Bl_p^M), L_{2k}, l_q^N) := \sup_{y \in L_{2k}} \inf_{x \in Bl_p^M} \|Ax - y\|_q \leq \lambda$$

Proof $\forall (z, \lambda) \in \mathrm{SP}_{2k}(A, p, q)_c = \{(x, \lambda) \in \mathrm{SP}(A, p, q) : S_c(x) = 2k\}$. By Definition 3 and Lemma 1, we have $S_c(Az) = S_c(z) = 2k$. Without loss of gerenality, we only consider the case of $S^+(Az) = S^+(z) = 2k$.

From $S_c(z) = 2k$ and $S^+(z) = 2k$, there exists $0 = i_0 < i_1 < i_2 < \cdots < i_{2k} < i_{2k+1} = M$

such that $z_j(-1)^{r+1} \geq 0$ $(or \leq 0)$, $i_{r-1}+1 \leq j \leq i_r$, $r=1, \cdots, 2k+1$, with the additional proviso that $z_j = 0$ implies $j = i_r$ for some r (If $z_{2k+1} = 0$ then $z_{2k} \neq 0$ and $i_{2k+1} = i_{2k}+1 = M$). Thus for $r = 1, \cdots, 2k$, either (a). $z_{i_r} z_{i_{r}+1} < 0$, or (b). $z_{i_r} = 0$, $z_{i_r-1} z_{i_r+1} < 0$.

We define vectors $\{e^r\}_{r=1}^{2k} \subset \mathbf{R}^M$, where

$$(e^r)_i = \begin{cases} |z_i|^{-(p-1)}, & i = i_l, i_l+1 \\ 0, & \text{others } i \end{cases} \quad \text{if } z_{i_l} z_{i_l+1} < 0;$$

$$(e^r)_i = \delta_{i i_l} \text{ if } z_{i_l} = 0.$$

$l = 1, \cdots, 2k$, then we have $(e^r, (z)_{(p)}) = 0$, $r = 1, \cdots, 2k$.

Similarly, we can construct $\{f^r\}_{r=1}^{2k} \subset \mathbf{R}^M$ such that $(f^r, Az) = 0$, $r = 1, \cdots, 2k$.

Set

$$\triangle := \begin{vmatrix} (f^1, Ae^1) & (f^1, Ae^2) & \cdots & (f^1, Ae^{2k}) \\ (f^2, Ae^1) & (f^2, Ae^2) & \cdots & (f^2, Ae^{2k}) \\ \cdots & \cdots & \cdots & \cdots \\ (f^{2k}, Ae^1) & (f^{2k}, Ae^2) & \cdots & (f^{2k}, Ae^{2k}) \end{vmatrix}.$$

If $\triangle = 0$, then, for $\epsilon > 0$ small enough we have

$$\triangle_\epsilon := \begin{vmatrix} (f^1, Ae^1) + \epsilon & (f^1, Ae^2) & \cdots & (f^1, Ae^{2k}) \\ (f^2, Ae^1) & (f^2, Ae^2) + \epsilon & \cdots & (f^2, Ae^{2k}) \\ \cdots & \cdots & \cdots & \cdots \\ (f^{2k}, Ae^1) & (f^{2k}, Ae^2) & \cdots & (f^{2k}, Ae^{2k}) + \epsilon \end{vmatrix} \neq 0,$$

since $\triangle_\epsilon$ is a polyminal with respect to ϵ with the root 0.

Without loss of gerenality, we can assume that $\triangle \geq 0$, and $\triangle_\epsilon > 0$ as $\triangle = 0$.

Now, we construct the $N \times M$ matrix $D = (d_{ij}) \in \mathbf{M}^{N \times M}$, by

1) If $\triangle \neq 0$, then

$$d_{ij} := \begin{vmatrix} (f^1, Ae^1) & \cdots & (f^1, Ae^{2k}) & (f^1, a^j) \\ \cdots & \cdots & \cdots & \cdots \\ (f^{2k}, Ae^1) & \cdots & (f^{2k}, Ae^{2k}) & (f^{2k}, a^j) \\ (Ae^1)_i & \cdots & (Ae^{2k})_i & a_{ij} \end{vmatrix} \cdot \triangle^{-1}.$$

2) If $\triangle = 0$, then

$$d_{ij} := \begin{vmatrix} (f^1, Ae^1) + \epsilon & (f^1, Ae^2) & \cdots & (f^1, Ae^{2k}) & (f^1, a^j) \\ (f^2, Ae^1) & (f^2, Ae^2) + \epsilon & \cdots & (f^2, Ae^{2k}) & (f^2, a^j) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (f^{2k}, Ae^1) & (f^{2k}, Ae^2) & \cdots & (f^{2k}, Ae^{2k}) + \epsilon & (f^{2k}, a^j) \\ (Ae^1)_i & (Ae^2)_i & \cdots & (Ae^{2k})_i & a_{ij} \end{vmatrix} \cdot \triangle_\epsilon^{-1},$$

where a^j denotes the ith column of A, $(Ae^l)_i$ denotes the ith component of Ae^l. Thus $\forall y \in \mathbf{R}^M$ we have

$$Dy = \begin{vmatrix} (f^1, Ae^1) & \cdots & (f^1, Ae^{2k}) & (f^1, Ay) \\ \cdots & \cdots & \cdots & \cdots \\ (f^{2k}, Ae^1) & \cdots & (f^{2k}, Ae^{2k}) & (f^{2k}, Ay) \\ Ae^1 & \cdots & Ae^{2k} & Ay \end{vmatrix} \cdot \triangle^{-1},$$

or

$$Dy = \begin{vmatrix} (f^1, Ae^1) + \epsilon & (f^1, Ae^2) & \cdots & (f^1, Ae^{2k}) & (f^1, Ay) \\ (f^2, Ae^1) & (f^2, Ae^2) + \epsilon & \cdots & (f^2, Ae^{2k}) & (f^2, Ay) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (f^{2k}, Ae^1) & (f^{2k}, Ae^2) & \cdots & (f^{2k}, Ae^{2k}) + \epsilon & (f^{2k}, Ay) \\ Ae^1 & Ae^2 & \cdots & Ae^{2k} & Ay \end{vmatrix} \cdot \triangle_\epsilon^{-1}.$$

Let $Dy = Ay - P_{2k}y$. Then $P_{2k} = (p_{ij}) \in \mathbf{M}^{N \times M}$, rank $P_{2k} = 2k$,

$$p_{ij} = a_{ij} - d_{ij} = \sum_{r=1}^{2k} b_{rj} (Ae^r)_i \quad i = 1, \cdots, N, \; j = 1, \cdots, M,$$

where b_{rj} is constant $(r = 1, \cdots, 2k;\; j = 1, \cdots, M)$.

We may easily prove some properties of D as follows (see [3]).

1) range$P_{2k} \le 2k$, rangeP_{2k} denotes the dimension of range of P_{2k};

2) $Dz = Az$, i.e., $P_{2k}z = 0$;

3) $\operatorname{sgn}(Az)_i d_{ij} \operatorname{sgn} z_j \ge 0 \quad i = 1, \cdots, N, \; j = 1, \cdots, M$;

4) $\sum_{i=1}^{N} d_{is} \left|(Dz)_i\right|^{q-1} \operatorname{sgn}(Dz)_i = \lambda^q \left|z_s\right|^{p-1} \operatorname{sgn} z_s, \; s = 1, \cdots, M$.

Set $D_+ = (|d_{ij}|) \in \mathbf{M}^{N \times M}$. By 3), we have

$$\max_{\|y\|_p = 1} \|Dy\|_q = \max_{\|y\|_p = 1} \|D_+ y\|_q := \mu$$

then there exists $z^* \in \mathbf{R}^M$ such that $\mu = \|D_+ z^*\|_q$, $\|z^*\|_p = 1$.

Since $D_+ \ge 0$ (i.e. $|d_{ij}| \ge 0$), we obtain $z^* = (z_1^*, \cdots, z_M^*) \ge 0$ (or ≤ 0) i.e. z_i^*, $i = 1, \cdots, M$ keep the same sign. We only consider the case of $z^* \ge 0$, then (z^*, μ) is a spectrum couples of D^+, i.e.

$$\sum_{i=1}^{N} |d_{ir}| \left|(D_+ z^*)_i\right|^{q-1} = \mu^q (z_r^*)^{p-1}, \; r = 1, \cdots, M. \tag{1}$$

From 3) and 4), for $u = (|z_1|, \cdots, |z_M|)$, we have

$$\sum_{i=1}^{N} |d_{ir}| \left|(D_+ u)_i\right|^{q-1} = \lambda^q |z_r|^{q-1}, \; r = 1, \cdots, M. \tag{2}$$

Thus $\lambda = \|D_+ u\|_q$, i.e. $\lambda \le \mu$.

We shall prove $\lambda \ge \mu$.

By the definition of d_{ij}, we have

(i) when $\triangle \ne 0$, if $z_r = 0$, then $|d_{ir}| = 0$ $i = 1, \cdots, N$. From (1) we have $z_r^* = 0$.

(ii) When $\triangle = 0$, for sufficient small $\epsilon > 0$, $\triangle_\epsilon \ne 0$, we have $|d_{ij}| \ne 0$, $i = 1, \cdots, N$, $j = 1, \cdots, M$.

Also, because $\|u\|_p = \|x\|_p = 1$, therefore $(D_+ u)_i > 0$, $i = 1, \cdots, M$. From (2), we have $|z_s| \ne 0$, $s = 1, \cdots, M$.

By the above discussions, there exists a smallest number γ (γ finite) such that $\gamma |z_s| \ge z_s^*$ for all s (From $\|u\|_p = \|z^*\|_p = 1 \Rightarrow \gamma \ge 1$). Thus

$$\gamma^{q-1} |(D_+ u)_i|^{q-1} \ge |(D_+ z^*)_i|^{q-1}, \quad i = 1, \cdots, N,$$

and therefore $\gamma^{q-1} \sum_{i=1}^{N} |d_{is}| |(D_+ u)_i|^{q-1} \ge \sum_{i=1}^{N} |d_{is}| |(D_+ z^*)_i|^{q-1}$.

From (1) and (2) we obtain

$$\gamma^{q-1} \lambda^q |z_s|^{p-1} \ge \mu^q (z_s^*)^{p-1}, \quad s = 1, \cdots, M.$$

Thus

$$\gamma^{\frac{q-1}{p-1}} \left(\frac{\lambda}{\mu}\right)^{\frac{q}{p-1}} |z_s| \ge z_s^*, \quad s = 1, \cdots, M.$$

From the definition of γ, it follows that $\lambda \ge \mu$. Thus $\lambda = \mu$.

Set $L_{2k} := \operatorname{span}\{Ae^1, Ae^2, \cdots, Ae^{2k}\}$. Then

$$d(A(Bl_p^M), L_{2k}, l_q^N) = \sup_{x \in A(Bl_p^M)} \inf_{y \in L_{2k}} \|x - y\|_q \le \sup_{z \in Bl_p^M} \|Az - P_{2k}z\|_q \ (P_{2k}z \in L_{2k})$$

$$= \sup_{z \in Bl_p^M} \|Dz\|_q = \max_{\|z\|_p = 1} \|D_+ z\|_q = \lambda$$

This proves Lemma 4. □

The Proof of Theorem 2 By Lemma 4 and the definition of $d_{2k}(A, l_p^M, l_q^N)$, $\forall (z, \lambda) \in SP_{2k}(A, p, q)_c$, we have

$$d_{2k}(A, l_p^M, l_q^N) \le d(A(Bl_p^M), L_{2k}, l_q^N) \le \lambda$$

From the proof of Lemma 4 and the definition of $\delta_{2k}(A, l_p^M, l_q^N)$, we obtain

$$\delta_{2k}(A, l_p^M, l_q^N) \le \sup_{z \in Bl_p^M} \|Az - P_{2k}z\|_q \le \lambda$$

Thus

$$d_{2k}(A, l_p^M, l_q^N) \le \lambda_{2k} \text{ and } \delta_{2k}(A, l_p^M, l_q^N) \le \lambda_{2k},$$

where $\lambda_{2k} = \inf\{\lambda: (x, \lambda) \in SP_{2k}(A, p, q)_c\}$.

From [4], we have

$$\delta_{2k}(A, l_p^M, l_q^N) \ge d_{2k}(A, l_p^M, l_q^N), \quad d^{2k}(A, l_p^M, l_q^N) \ge b_{2k}(A, l_p^M, l_q^N);$$

and $d^{2k}(A^T, l_q^{N'}, l_p^{M'}) = d_{2k}(A, l_p^M, l_q^N)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, A^T is the transpose of A. And we have already known that $d_{2k}(A, l_p^M, l_q^N) \geq \lambda_{2k}$. Hence

$$d_{2k}(A, l_p^M, l_q^N) = \delta_{2k}(A, l_p^M, l_q^N) = \lambda_{2k} = d^{2k}(A^T, l_{q'}^N, l_{p'}^M) \geq b_{2k}(A, l_p^M, l_q^N).$$

Since $SP_{2k}(A, p, q)_c$ is a closed subset, therefore

$$\lambda_{2k} \in \{\lambda: (x, \lambda) \in SP_{2k}(A, p, q)_c\}.$$

Let x^* is the corresponding spectrum vector, then for (x^*, λ_{2k}), in the proof of Lemma 4 the corresponding linear subspace $L_{2k}^* = \mathrm{span}\{Ae_*^1, \cdots, Ae_*^{2k}\}$ and the matrix $P_{2k}^* = (p_{ij}^*)_{N \times M}$ (where $p_{ij}^* \sum_{r=1}^{2k} b_{rj}^* (Ae_*^r)_i$, $i = 1, \cdots, N$, $j = 1, \cdots, M$) are the optimal subspace for d_{2k} and the optimal matrix for δ_{2k}, respectively. This proves Theorem 2. □

Remark 1 The result of Theorem 2 holds for $q = 1$, $1 < p < +\infty$, too (see [5]).

Theorem 3 *Let $A \in \mathbf{M}^{N \times M}$, M is odd, $N > M = \mathrm{rank}A$, $1 \leq q \leq p < +\infty$ $(p \neq 1)$. If $A \in CVD(N \times M)$, then*

$$d_{2k}(A, l_p^M, l_q^N) = \delta_{2k}(A, l_p^M, l_q^N) = d^{2k}(A^T, l_{q'}^N, l_{p'}^M) \geq b_{2k}(A, l_p^M, l_q^N).$$

Proof By Lemma 1, we need only to prove the results for $A \in SC_{2r-1}(N \times M)$, $r = 1, \cdots, \frac{M+1}{2}$.

From Lemma 2, there exists $\{A_\sigma\} \subset SSC_{2r-1}(N \times M)$, $r = 1, \cdots, \frac{M+1}{2}$, such that $\lim_{\sigma \to \infty} A_\sigma = A$.

By Theorem 2 and Remark 1, we obtain

$$d_{2k}(A_\sigma, l_p^M, l_q^N) = \delta_{2k}(A_\sigma, l_p^M, l_q^N) = d^{2k}(A_\sigma^T, l_{q'}^N, l_{p'}^M) \geq b_{2k}(A_\sigma, l_p^M, l_q^N),$$
$$1 \leq q \leq p < +\infty \ (p \neq 1).$$

From Lemma 3, it follows that

$$d_{2k}(A_\sigma, l_p^M, l_q^N) \longrightarrow d_{2k}(A_\sigma, l_p^M, l_q^N) \ (\sigma \to \infty)$$
$$\delta_{2k}(A_\sigma, l_p^M, l_q^N) \longrightarrow \delta_{2k}(A_\sigma, l_p^M, l_q^N) \ (\sigma \to \infty)$$
$$d^{2k}(A_\sigma^T, l_{q'}^N, l_{p'}^M) \longrightarrow d^{2k}(A_\sigma^T, l_{q'}^N, l_{p'}^M) \ (\sigma \to \infty)$$
$$b_{2k}(A_\sigma, l_p^M, l_q^N) \longrightarrow b_{2k}(A_\sigma, l_p^M, l_q^N) \ (\sigma \to \infty)$$

Thus

$$d_{2k}(A, l_p^M, l_q^N) = \delta_{2k}(A, l_p^M, l_q^N) = d^{2k}(A^T, l_{q'}^N, l_{p'}^M) \geq b_{2k}(A, l_p^M, l_q^N).$$

This proves Theorem 3. □

Remark 2 For $p = +\infty$, $1 \leq q \leq +\infty$, we can obtain the following result

$$d_{2k}(A, l_\infty^M, l_q^N) = d^{2k}(A, l_\infty^M, l_q^N) = \delta_{2k}(A, l_\infty^M, l_q^N).$$

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References

[1] A. P. Buslaev, *On a variatoional description of the spectrum of total positive matrices and extremal problems in approximation theory*, Mat. Zametki, **47**: **1**(1990), 39-46.

[2] S. Karlin, *Total positivity*, Vol. 1, Standford University Press 1968.

[3] A. Pinkus, *Some extremal problems for strictly totally positive matrices*, Linear Algebra and its Appl., **64**(1985), 141-156.

[4] A. Pinkus, *n-width in approximation theory*, Springer-verlag, Berlin, 1985.

[5] Sun Yongsheng, *The theory of approximation (I)* (*In Chinese*). Beijing Normal University Press, 1990.

关于CVD 矩阵的 n- 宽度

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摘　要

本文证明了 $d_{2k}=\delta_{2k}=d^{2k}\geq b_{2k}$, **其中** d_{2k}, δ_{2k}, b_{2k}**分别表示**$A\,(B\,l_p^M)$**在** l_q^N **下的** Kolmogorov, **线性**, Bernstein $2k$-**宽度**, d^{2k}**表示**$A^T(B\,l_{q'}^N)$**在** $l_{p'}^M$**下的** Gelfand $2k$-**宽度**, A **是一个**$N\times M$ **的** CVD **矩阵**($N>M=\mathrm{rank}\,A$, M **是奇数**), $B\,l_p^M$ **表示** l_p^M **中的单位球**, A^T **是**A **的转置**, $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{q}+\frac{1}{q'}=1$ $(1\leq q\leq p<+\infty, p\neq 1)$.