Journal of Mathematical Research & Exposition
Vol. 17, No. 4, 479- 488, Nov. 1997

Nearly Strict Convexity and Best Approximation*

Nan Chaoxun Song Shoubai
(Dept. of Math., Anhui Normal University, Wuhu 241000)

Abstract In this paper, we study nearly strict convexity and the best approximation in nearly strictly convex spaces. We prove that a Banach space X is nearly strictly convex if and only if all of the subspaces of X are compact-semi-Chebyshev subspaces. We also show that Theorem 6 in [10] is false.

Keywords nearly strict convex, compact-semi-Chebyshev spaces, nearly extreme points, metric projection.

Classification AMS(1991) 46B20, 41A65/CCL O177.3

In 1960, Singer introduced the k-strictly convex Banach spaces and proved that a Banach space X is $(k+1)$-strictly convex if and only if all of the subspaces of X are k-semi-Chebyshev subspaces. In 1988, Sekowski and Stachura introduced the nearly strictly convex Banach spaces with the Kuratowski measure of non-compactness. In the present paper, we discuss the best approximation in nearly strictly convex Banach spaces. First we give some equivalent conditions for nearly strict convexity and point out that nearly strict convexity may be lifted to the Banach space $l^p(X_i)$. Then we introduce the compact-semi-Chebyshev space to prove that a Banach space X is nearly strictly convex if and only if every subspace of X is compact-semi-Chebyshev subspace. Finally, we discuss continuity of the metric projection in locally nearly uniformly convex Banach spaces.

1.

Let X be a Banach space. We will denote the closed unit ball and unit sphere of X by $U(X)$ and $S(X)$, respectively.

For a nonempty bounded set A in X, the Kuratowski measure of non-compactness of A is

$$\alpha(A) = \inf\{r: A \text{ is covered by a finite family of sets of diameter less than } r\}.$$

It is clear that $\alpha(A) = 0$ if and only if A is relatively compact.

Definition 1[7] *A Banach space X is said to be nearly strictly convex (NSC) if the unit sphere*

* Received Sep. 13, 1994.

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

$S(X)$ *contains no convex subset* A *with* $\alpha(A) > 0$.

Equivalently, we have

Definition 1 [11] *A Banach space* X *is said to be nearly strictly convex if every convex subset of* $S(X)$ *is relatively compact.*

Recall that a Banach space X is said to be k-strictly convex if for any $k+1$ elements $x_1, x_2, \cdots, x_{k+1}$ in $S(X)$ whenever $\|x_1 + x_2 + \cdots + x_{k+1}\| = k+1$, $x_1, x_2, \cdots, x_{k+1}$ are linearly dependent. In [8], it is proved that X is k-strictly convex if and only if the unit sphere $S(X)$ contains no convex subset of dimension bigger than $k-1$. Hence k-strictly convex space is nearly strictly convex, but the converse is not true.

Definition 2 *Let* X *be a Banach space. A point* $x \in S(X)$ *is called a nearly extreme point of* $U(X)$ *if there does not exist noncompact closed convex subset* A *in* $S(X)$ *such that* $x \in A$.

Clearly, X is nearly strictly convex if and only if each point of $S(X)$ is a nearly extreme point of $U(X)$.

Theorem 1 *Let* X *be a Banach space and* $x_0 \in S(X)$. *Then* x_0 *is a nearly extreme point of* $U(X)$ *if and only if whenever* $f \in S(X^*)$ *and* $f(x_0) = 1$, *the set* $A_f = \{x \in S(X): f(x) = 1\}$ *is compact.*

Proof Suppose that x_0 is a nearly extreme point of $U(X)$, $f \in S(X^*)$ and $f(x_0) = 1$, then $x_0 \in A_f$. Obviously, $A_f \subset S(X)$ and A_f is a closed convex set. So A_f is compact.

Conversely, suppose that x_0 is not a nearly extreme point of $U(X)$, then there exists a noncompact closed convex subset C in $S(X)$ such that $x_0 \in C$. Since $C \cap U^0(X) = \emptyset$, where $U^0(X) = \{x \in X: \|x\| < 1\}$, by the separation theorem, there exists f in $S(X^*)$ such that

$$\sup_{x \in U(X)} f(x) \le \inf_{x \in C} f(x).$$

It follows that for every y in C we have

$$\|f\| = \sup_{x \in U(X)} f(x) \le \inf_{x \in C} f(x) \le f(y) \le \|f\| = 1.$$

Therefore $f(y) = 1$. Thus $C \subset A_f$, so A_f is not compact. □

Theorem 2 *Let* X *be a Banach space. Then the following statements are equivalent.*

(1) X is nearly strictly convex.

(2) For every x in X and each $\delta > 0$, the set $\{y \in X: \|x - y\| = \delta\}$ does not contain noncompact closed convex subset.

(3) For every f in $S(X^*)$, the set $A_f = \{x \in S(X): f(x) = 1\}$ is compact.

(4) For any sequence $\{x_n\}$ in $S(X)$, if $\|x_1 + x_2 + \cdots + x_k\| = k$ for all integer $k \ge 1$, then $\{x_n\}$ is relatively compact.

Proof (1)$\Leftrightarrow$(2) It is trivial.

(1)$\Leftrightarrow$(3) It follows from Theorem 1.

(1)$\Rightarrow$(4) Suppose that X is nearly strictly convex. Let $\{x_n\}$ be a sequence in $S(X)$ and $\|x_1 + x_2 + \cdots + x_k\| = k$ for all integer $k \ge 1$. Choose f_k, $k = 1, 2, \cdots$, such that $f_k(\frac{1}{k}(x_1 + x_2 +$

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

$\cdots + x_k)) = 1$. It is easy to see that $f_k(x_i) = 1$ for $1 \le i \le k$. Since $U(X^{\pm})$ is weak* compact, $\{f_k\}$ has weak* cluster points. Let f be a weak* cluster point of $\{f_k\}$. Now by $f_k(x_n) = 1$ for all $k \ge n$, we have $f(x_n) = 1$. Therefore $f \in S(X^*)$ and $x_n \in A_f$. By the equivalency of (1) and (3), A_f is a compact set, hence $\{x_n\}$ is relatively compact.

(4) $\Rightarrow$ (3) Suppose that $f \in S(X^*)$ and $\{x_n\} \subset A_f$. Then for every k, $f(\frac{1}{k}(x_1 + x_2 + \cdots + x_k)) = \frac{1}{k}(f(x_1) + f(x_2) + \cdots + f(x_k)) = 1$. Obviously $f(\frac{1}{k}(x_1 + x_2 + \cdots + x_k)) \le \|f\| \|\frac{1}{k}(x + x_2 + \cdots + x_k)\| = \|\frac{1}{k}(x_1 + x_2 + \cdots + x_k)\| \le 1$, which implies that $\|x_1 + x_2 + \cdots + x_k\| = k$ for all k. Thus $\{x_n\}$ is relatively compacet, so A_f is a compact set. □

Theorem 3 *Let X be a Banach space. Then X is nearly strictly convex if and only if every separable subspace of X is nearly strictly convex.*

Proof We need only to prove the sufficiency. Suppose that X is not nearly strictly convex, then there exists a noncompact closed convex set in $S(X)$. Thus we can pick out a linearly independent set $\{x_n\}_{n=1}^{\infty}$ from C such that $\{x_n\}_{n=1}^{\infty}$ is not relatively compact. Let $M = \overline{\text{span}}(\{x_n\}_{n=1}^{\infty})$, then M is a separable subspace of X and $\overline{\text{co}}(\{x_n\}_{n=1}^{\infty}) \subset S(M)$. Obviously, $\overline{\text{co}}(\{x_n\}_{n=1}^{\infty})$ is not compact, which contradicts the fact that M is nearly strictly convex. □

Let $\{X_i\}$ be a sequence of Banach spaces and $1 \le p < \infty$. The direct sum of these spaces in the sense of l^p is defined as follows

$$l^p(X_i) = \{x = (x_1, \cdots, x_i, \cdots): x_i \in X_i,\ i = 1, 2, \cdots, \text{ and } \sum_{i=1}^{\infty} \|x_i\|^p < \infty\},$$

and

$$\|x\| = (\sum_{i=1}^{\infty} \|x_i\|^p)^{1/p}.$$

It is known that $l^p(X_i)$ is a Banach space and $(l^p(X_i))^* = l^q(X_i^*)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, for $x = (x_1, \cdots, x_i, \cdots)$ in $l^p(X_i)$, $f = (f_1, \cdots, f_i, \cdots)$ in $l^q(X_i^*)$, we have

$$f(x) = \sum_{i=1}^{\infty} f_i(x_i).$$

Furthermore, if every X_i is strictly convex then $l^p(X_i)$ $(1 < p < \infty)$ is strictly convex. For nearly strict convexity, we have the similar result

Lemma 1[5] *Let $\{X_i\}$ be a sequence of Banach space, and $1 < p < \infty$. Then a subset K in $l^p(X_i)$ is compact if and only if*

(I) K is closed and bounded;

(II) given any $\epsilon > 0$, there exists an $n_0 = n_0(\epsilon)$ (depending only on ϵ) such that $(\sum_{i=n+1}^{\infty} \|x_i\|^p)^{1/p} < \epsilon$ for all $x = (x_1, \cdots, x_i, \cdots) \in K$ whenever $n \ge n_0$;

(III) Let p_i: $l^p(X_i) \to X_i$ be a mapping with $p_i(x) = x_i$ for all $x = (x_1, \cdots, x_i, \cdots)$ in $l^p(X_i)$, then $p_i(K)$ is compact for all $i \ge 1$.

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

Theorem 4 *Let $\{X_i\}$ be a sequence of nearly strictly convex Banach spaces and $1 < p < \infty$. Then $l^p(X_i)$ is nearly strictly convex.*

Proof Choose any f in $(l^p(X_i))^*$, $f = (f_1, \cdots, f_i, \cdots)$, where $f_i \in X_i^*$, and $\|f\|_q = (\sum_{i=1}^{\infty} \|f_i\|^q)^{1/q} = 1$, $1/p + 1/q = 1$. By Theorem 2, we need only to prove that the set $A_f = \{x \in S(l^p(X_i)) : f(x) = 1\}$ is compact.

(I) It is evident that A_f is closed and bounded.

(II) For every $x \in A_f$, $x = (x_1, \cdots, x_i \cdots)$, we have

$$1 = f(x) = \sum_{i=1}^{\infty} f_i(x_i) \le \sum_{i=1}^{\infty} \|f_i(x_i)\| \le \sum_{i=1}^{\infty} \|f_i\| \|x_i\| \le \|f\|_q \|x\| = 1. \qquad (*)$$

Now, let $b = (b_1, \cdots, b_i, \cdots)$, $a = (a_1, \cdots, a_i, \cdots)$, where $b_i = \|f_i\|$, $a_i = \|x_i\|$, then $b \in l^q$, $a \in l^p$, $\|b\| = \|a\| = 1$ and $b(a) = 1$.

Since l^p is strictly convex, a is the only point at which b achieves its norm. Thus the norm of the ith-coordinate of any element of A_f is a_i.

Now for any $\epsilon > 0$, by $(\sum_{i=1}^{\infty} a_i^p)^{1/p} = 1$, there is an $n_0 = n_0(\epsilon)$, such that

$$(\sum_{i=n+1}^{\infty} a_i^p)^{1/p} < \epsilon, \qquad n \ge n_0.$$

It follows that for all $x = (x_1, \cdots, x_i, \cdots)$ in A_f, we have

$$(\sum_{i=n+1}^{\infty} \|x_i\|^p)^{1/p} < \epsilon, \qquad n \ge n_0.$$

(III) By $(*)$, for every $x = (x_1, \cdots, x_i, \cdots)$ in A_f and every i, we have

$$f_i(x_i) = \|f_i\| \|x_i\| = b_i a_i.$$

If $b_i = 0$, then $f_i = 0$, so by $(*)$ $x_i = 0$. Therefore $p_i(A_f) = \{0\}$ which is a compact set. If $b_i \ne 0$, then $f_i \ne 0$ and $x_i \ne 0$. For convenience, let $g_i = (1/b_i) f_i$, then

$$g_i(x_i/a_i) = 1, \ g_i \in X_i^* \text{ and } \|g_i\| = 1.$$

Thus $g_i \in S(X_i^*)$ and $x_i/a_i \in A_{g_i}$. Since X_i is nearly strictly convex, A_{g_i} is a compact set. Consequently $p_i(A_f)$ is a compact subset of X_i because of $p_i(A_f)$ is exactly the set $a_i A_{g_i}$.

To sum up, by Lemma 1, $l^p(X_i)$ is nearly strictly convex. □

Corollary *Let $\{X_i\}$ be a sequence of Banach spaces and $\{k_i\}$ be a sequence of positive integers. If X_i is k_i-strictly convex for each i and $1 < p < \infty$, then $l^p(X_i)$ is nearly strictly convex.*

Remark[10] Now we point out that the following result is false.

Theorem Let $\{X_i\}$ be a sequence of Banach spaces and let $\{k_i\}$ be a sequence of positive integers such that $k = \sup\{k_1, k_2, \cdots\} < \infty$. If X_i is k_i-strictly convex for each i and $1 < p < \infty$, then $l^p(X_i)$ is k-strictly convex.

In fact, under the conditions of this Theorem we can only conclude that $l^p(X_i)$ is nearly strictly convex.

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

Example Let $X = \{(x, y): x, y \in R$ and $\|(x, y)\| = |x| + |y|\}$ and $X_i = X$ for each i. Obviously, X_i is 2-strictly convex, but $l^2(X_i)$ is not 2-strictly convex. In fact, take z_1, z_2 and z_3 in $l^2(X_i)$ such that

$$z_1 = ((1, 0), (0, 1), 0, \cdots),$$
$$z_2 = ((1/2, 1/2), (1, 0), 0,$$
$$z_3 = ((0, 1), (1/2, 1/2), 0, \cdots).$$

Obviously, $\|z_1\| = \|z_2\| = \|z_3\| = \sqrt{2}$, $\|z_1 + z_2 + z_3\| = 3\sqrt{2}$. However, z_1, z_2, z_3 is not linearly dependent. Since $a_1z_1 + a_2z_2 + a_3z_3 = 0$ implies $a_1 + a_2/2 = 0$, $a_3 + a_2/2 = 0$ and $a_1 + a_3/2 = 0$, so $a_1 = a_2 = a_3 = 0$.

Definition 3 [7] *Let X be a Banach space. A point $x \in S(X)$ is called a nearly smooth point of X if $\alpha(\{f \in S(X^*): f(x) = 1\}) = 0$. X is said to be nearly smooth if every point of $S(X)$ is a nearly smooth point of X.*

Following the proof of Theorem 4, we have

Theorem 5 *If $\{X_i\}$ is a sequence of nearly smooth Banach spaces and $1 < p < \infty$, then $l^p(X_i)$ is nearly smooth.*

2

Let X be a Banach space and M be a subspace of X. For x in X, set

$$P_M(x) = \{z \in M: \|x - z\| = \inf_{x \in M} \|x - y\|\}$$

$P_M(x)$ is the set of all elements of the best approximation to x from M. Obviously, $P_M(x)$ is a bounded convex set, but $P_M(x)$ may be empty. The subspace M is called a proximinal subspace of X if for every $x \in X$, $P_M(x)$ is nonempty, and M is called a Chebyshev subspace of X if for every $x \in X$, $P_M(x)$ is a singleton.

Let $\dim P_M(x)$ denote the dimension of $P_M(x)$. We say that M is a k-semi-Chebyshev subspace (respectively a k-Chebyshev subspace), where k is a nonnegative integer, if for any x in X, we have $-1 \leq \dim P_M(x) \leq k$ (respectively, $0 \leq \dim P_M(x) \leq k$); a 0-semi-Chebyshev subspace is also called a semi-Chebyshev subspace, and 0-Chebyshev subspace is the Chebyshev subspace. I Singer had proved that X is $(k+1)$-strictly convex if and only if all subspaces of X are k-semi-Chebyshev subspaces [8]. Particularly, X is strictly convex if and only if for each subspace M of X and each $x \in X$, $P_M(x)$ contains at most one element.

Now, we study the relations between nearly strict convexity and the best approximation.

Definition 4 *Let X be a Banach space. A subspace M of X is said to be a compact-semi-Chebyshev subspace of X, if for every $x \in X$, $P_M(x)$ is a compact set; M is said to be a compact-Chebyshev subspace of X, if for every $x \in X$, $P_M(x)$ is nonempty and compact.*

Obviously, k-semi-Chebyshev space (respectively k-Chebyshev space) is compact-semi-Chebyshev space (respectively compact-Chebyshev space).

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

Lemma 2 [8] *Let X be a Banach space and M be a subspace of X. Let $x \in X \setminus M$, $y \in M$. Then $y \in P_M(x)$ if and only if there exists f in $S(X^*)$, such that*

$$f(m) = 0, \qquad m \in M$$

and

$$f(x - y) = \|x - y\|.$$

Lemma 3 [8] *Let X be a Banach space and M be a subspace of X, if $x \in X \setminus M$, and $A \subset M$, then $A \subset P_M(x)$ if and only if there exists f in $S(X^*)$, such that*

$$f(m) = 0, \qquad m \in M,$$

$$f(x - y) = \|x - y\|, \qquad y \in A.$$

Theorem 6 *Let X be a Banach space and M be a subspace of X. Then M is compact semi-Chebyshev subspace if and only if there does not exist $x \in X$, $f \in S(X^*)$, and a sequence $\{x_n\}$ in M which is non-relatively compact and linearly independent, such that*

(I) $f(m) = 0, \qquad m \in M,$

and

(II) $f(x) = \|x\| = \|x - x_n\|, \qquad n = 1, 2, \cdots.$

Proof Suppose on the contray that there exist $f \in S(X^*)$, $x \in X$ and a non-relatively compact and linearly independent set $\{x_n\}$ in M such that (I) and (II) hold. Then $x \notin M$, since otherwise, by (I), $f(x) = 0$, and by (II), $x = 0$. Again by $0 = f(x - x_1) = \|x - x_1\|$, we have $x - x_1 = 0$, therefore $x_1 = 0$ which contradicts the linearly independency of $\{x_n\}$. By Lemma 2, we obtain that $\{x_n\} \subset P_M(x)$. Thus, $P_M(x)$ is not compact which contradicts with the fact that M is a compact-semi-Chebyshev subspace.

Conversely, suppose that M is not compact-semi-Chebyshev subspace, then there exists y in $X \setminus M$ such that $P_M(y)$ is not compact (note that for any y in M, $P_M(y)$ is compact). Thus we can choose a non- relatively compact and linearly independent set $\{y_n\}$ in $P_M(y)$. Set

$$x = y - y_1,\ x_1 = y_2 - y_1, \cdots, x_n = y_{n+1} - y_1, \cdots,$$

then $x \notin M$ and $P_M(x) = P_M(y - y_1) = P_M(y) - y_1 \supset \{0, x_1, x_2, \cdots\}$. Let $A = \{0, x_1, x_2, \cdots\}$, by Lemma 3, there is an f in $S(X^*)$, such that

$$f(m) = 0, \qquad m \in M,$$

and

$$f(x) = \|x\| = \|x - x_n\|, \qquad n = 1, 2, \cdots$$

which contradicts the assumption of sufficiency.

Theorem 7 *Let X be a Banach space, then X is nearly strictly convex if and only if all subspaces of X are compact-semi-Chebyshev subspaces.*

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

Proof Necessity. Let X be nearly strictly convex and M be a subspace of X. If M is not a compact-semi-Chebyshev space, then there exists x in X such that $P_M(x)$ is not compact. Since $P_M(x)$ is a convex set and for every $y \in P_M(x)$, $\|x - y\| = d(x, M) = \delta$ where $\delta > 0$, hence the set $\{z \in X: \|x - z\| = \delta\}$ contains the noncompact convex set $P_M(x)$. By the nearly strict convexity of X, this is impossible.

Sufficiency. For any f in $S(X^*)$, we need only to prove that $A_f = \{x \in S(X): f(x) = 1\}$ is compact.

Now suppose that A_f is not empty. Let $H = \{x \in X: f(x) = 1\}$, then $A_f \subset H$, $\inf_{x \in H}\|x\| = 1$ and for x in H, $\|x\| = 1$ if and only if $x \in A_f$. Take x_0 in A_f and set $M = H - x_0$. Then M is a maximal subspace of X, and M is the null space of f. Let $g = -f$, then for each $u \in A_f$, we have

$$g(-x_0 - (u - x_0)) = 1 = \|u\| = \|-x_0 - (u - x_0)\|.$$

By Lemma 2, $u - x_0 \in P_M(-x_0)$, which implies $A_f - x_0 \subset P_M(-x_0)$. Since M is a compact-semi-Chebyshev space, $P_M(-x_0)$ is compact, hence A_f is also compact. This proves that X is nearly strictly convex.

Remark In fact, we can prove that $P_M(-x_0) = A_f - x_0$. Let $y = u - x_0 \in M$ and $y \in P_M(-x_0)$, then, by Lemma 2, there exists $h \in S(X^*)$ such that

$$h(m) = 0, \quad m \in M,$$

and

$$h(-x_0 - y) = h(-x_0 - (u - x_0)) = \|-x_0 - (u - x_0)\| = \|u\|.$$

Since $u \in H$, $\|u\| \geq 1$. On the other hand, $h(-x_0 - y) = h(-x_0) \leq \|h\|\|-x_0\| = 1$. Hence $\|u\| = 1$. Therefore, $u \in A_f$, and this implies $y \in A_f - x_0$. Thus $P_M(-x_0) \subset A_f - x_0$. By the proof of Theorem 7, we obtain $P_M(-x_0) = A_f - x_0$.

Theorem 8 *Let X be a Banach space. Then X is reflexive and nearly strictly convex if and only if every subspace of X is compact-Chebyshev space.*

Proof This follows from Theorem 7 and the fact that X is reflexive if and only if every subspace of X is proximinal space.

Let $\{X_i\}$ be a sequence of Banach spaces, $X = l^p(X_i)$ and $1 < p < \infty$. Let M be a subspace of X and let

$$M_i = \{x_i \in X_i: x = (x_1, \cdots, x_i, \cdots) \in M\},$$

then M_i is a subspace of X_i. In [6], it was proved that M is a semi-Chebyshev (respectively, Chebyshev) subspace of X if and only if for each i, M_i is a semi-Chebyshev (respectively, Chebyshev) subspace of X_i. Similarly, we have the following

Theorem 9 *Let $X = l^p(X_i)$, $1 < p < \infty$. If M is a subspace of X and M_i is the same as above. Then*

(I) M is a compact-semi-Chebyshev subspace of X if and only if for each i, M_i is a compact-semi-Chebyshev subspace of X_i.

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

(II) M is a compact-Chebyshev subspace of X if and only if for each i, M_i is a compact-Chebyshev subspace of X_i.

3.

Finally, we study a class of nearly strictly convex spaces and continuity of metric projection.

In 1980, Huff introduced the nearly uniformly convex (NUC) Banach spaces which is an important generalization of uniformly convex spaces. Recently, Kutzarova and Bor-Luh Lin studied the localization of NUC, and defined the locally nearly uniformly convex (LNUC) Banach spaces.

Definition 5[3][4] *A Banch space X is said to be locally nearly uniformly convex (LNUC) if for every $\epsilon > 0$ and every x in $U(X)$. there exists $\delta = \delta(x, \epsilon) > 0$, such that for any sequence $\{x_n\}$ in $U(X)$ with* $\mathrm{sep}(x_n) > \epsilon$ *then* $\mathrm{co}(\{x\} \cup \{x_n\}) \cap (1 - \delta) \cup (X) \neq \emptyset$, *where* $\mathrm{sep}(x_n) = \inf_{n \neq m} \|x_n - x_m\|$

In [3], it was proved that every locally k-uniformly rotund (Lk-UR) space is LNUC. The Lk-UR spaces was introduced by F. Sullivan.[9]

Theorem 10 *If X is* LNUC, *then X is nearly strictly convex.*

Proof Assuming that X is not nearly strictly convex, then there exists f in $S(X^*)$ such that $A_f = \{x \in S(X): f(x) = 1\}$ is not compact. Therefore there exists a sequence $\{x_n\}$ in A_f and $\epsilon > 0$ such that $\mathrm{sep}(x_n) = \inf_{n \neq m} \|x_n - x_m\| > \epsilon$. Since A_f is a convex set, thus for any $\delta > 0$, $\mathrm{co}(\{x_1\} \cup \{x_n\}_{n=2}^{\infty}) \cap (1 - \delta) \cup (X) = \emptyset$, which contradicts the hypothesis of this Theorem.

Let M be a Chebyshev subspace of X, then for each x in X, $P_M(x)$ is a singleton. Set $y = P_M(x)$, then P_M is a operator from X to M. The operator P_M is called the metric projection on M.

Theorem 11 *Let X be* LNUC *and M be a Chebyshev subspace of X. Then the metric projection P_M is continuous.*

Proof Let $x, x_n \in X$ and $x_n \to x$. If $x \in M$, it is easy to see that $P_M(x_n) \to P_M(x)$. If $x \notin M$, then $d(x, M) > 0$ because M is a closed subspace. By the proof of Theorem 1 in [9, p632], we may assume that $\|x\| = 1$ and $P_M(x) = 0$ and we need only to show that $P_M(x_n) \to 0$. In fact, it is sufficient to show that $\{P_M(x_n)\}$ is relatively compact because if $\{P_M(x_n)\}$ has a subsequence which converges to y, then by uniqueness of the best approximation, $y = 0$. Moreover, it is evident that $\{P_M(x_n)\}$ is relatively compact if and only if $\{x - P_M(x_n)\}$ is relatively compact.

Suppose that $\{x - P_M(x_n)\}$ is not relatively compact, then there exist $\epsilon > 0$ and a subsequence of $\{x - P_M(x_n)\}$ -for convenience we denote it again by $\{x - P_M(x_n)\}$, such that

$$\mathrm{sep}(x - P_M(x_n)) = \mathrm{sep}(P_M(x_n)) = \inf_{n \neq m} \|P_M(x_n) - P_M(x_m)\| > \epsilon$$

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

Since

$$\|x\| \leq \|x - P_M(x_n)\| \leq \|x - x_n\| + \|x_n - P_M(x_n)\| \leq \|x - x_n\| + \|x_n\| \to 1,$$

hence $\|x - P_M(x_n)\| \to 1$.

For and integer $k \geq 1$, $c_0, c_1, \cdots, c_k \geq 0$, $c_0 + c_1 + \cdots + c_k = 1$ and $n_1, n_2, \cdots, n_k$, we have

$$\begin{aligned}&\|c_0 x + c_1(x - P_M(x_{n_1})) + \cdots + c_k(x - P_M(x_{n_k}))\| \\ &\leq c_0\|x\| + c_1\|x - P_M(x_{n_1})\| + \cdots + c_k\|x - P_M(x_{n_k})\|\end{aligned}$$

and

$$\begin{aligned}&\|c_0 x + c_1(x - P_M(x_{n_1})) + \cdots + c_k(x - P_M(x_{n_k}))\| \\ &= \|x - [c_1 P_M(x_{n_1}) + \cdots + c_k P_M(x_{n_k})]\| \geq \|x\| = 1,\end{aligned}$$

which implies that

$$\lim_{n_1, \cdots, n_k \to \infty} \|c_0 x + c_1(x - P_M(x_{n_1})) + \cdots + c_k(x - P_M(x_{n_k}))\| = 1.$$

Thus for any $\delta > 0$, there exist $N(\delta)$ such that for any integer $k \geq 1$ and $c_0, c_1, \cdots, c_k \geq 0$, $c_0 + c_1 + \cdots + c_k = 1$ whenever $n_1, \cdots, n_k \geq N(\delta)$, we have

$$\|c_0 x + c_1(x - P_M(x_{n_1})) + \cdots + c_k(x - P_M(x_{n_k}))\| > 1 - \delta$$

Therefore we have $\operatorname{sep}(x - P_M(x_n))_{n=m}^{\infty} > \epsilon$, and

$$\operatorname{co}(\{x\} \cup \{x - P_M(x_n)\}_{n=m}^{\infty}) \cap (1 - \delta) \cup (X) = \emptyset,$$

where $m \geq N(\delta)$. This contradicts the hypothesis that X is LNUC.

Lemma 4[6] *Let $X = l^p(X_i)$, $1 < p < \infty$ and let M be a Chebyshev subspace of X. If $M_i = \{x_i \in X: x = (x_1, \cdots, x_i, \cdots) \in M\}$ and P_{M_i} is the metric projection of X_i on M_i, then P_M is continuous if and only if for each i, P_{M_i} is continuous.*

Theorem 12 *Let $\{X_i\}$ be a sequence of LNUC spaces, $X = l^p(X_i)$ and $1 < p < \infty$. If M is a Chebyshev subspace of X, then P_M is continuous.*

Proof *Let $x^{(n)} = (x_1^{(n)}, \cdots, x_i^{(n)}, \cdots)$, $x = (x_1, \cdots, x_i, \cdots)$ in X and $x^{(n)} \to x$. Then for each i, $x_i^{(n)} \to x_i$ in X_i. By Theorem 2 in [6], M_i is a Chebyshev subspace of X_i, and by Theorem 11, P_{M_i} is continuous at point x_i. Using Lemma 4, we obtain that P_M is continuous at point x. This completes the proof of this Theorem.*

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

References

[1] J. Banas, *On drop property and nearly uniformly smooth Banach spaces*, Nonlinear Analysis, **14**(1990), 927 - 933.

[2] R. Huff, *Banach spaces which are nearly uniformly convex*, Rocky. Mountain J. Math., **10**(1980), 734-749.

[3] Kutzarova and Bor-Luh Lin, *Locally k- nearly uniformly convex Banach spaces*, (to appear)

[4] Bor-Luh Lin and Wenyao Zhang, *Some geometric properties related touniform convexity of Banach spaces*, Function spaces (Edwardsville, II, 1990) 281- 293, Lecture Notes in Pure and Appl Math., 36, Dekller, New York, 1992.

[5] J. E. Leonard, *Banach sequence spaces*, J. Math. Anal Appl, **54**(1976), 245- 265.

[6] Nan Chaoxun, *Chebyshev sets and continuity of metric projection in Banach sequence spaces*, Journal of Anhui Normal University (Natural Science), **16**: **4**(1993), 7- 11.

[7] T. Sekowski and A. Stachura, *Noncompact smoothness and noncompact convexity*, Atti Semi Mat Fis Univ. Modena, **36**(1988), 329- 338.

[8] I Singer, *Best approximation in normed linear spaces by elements of linear subspaces*, Springer Verlag, Berlin, 1970.

[9] F. Sullivan, *A generalization of uniformly round Banach spaces*, Cand. J. Math., **31**(1979), 628- 636.

[10] Zhuang Yadong, *On k- rotund complex normed linear spaces*, J. Math. Anal Appl, **174**(1993), 218-230.

近严格凸与最佳逼近

南朝勋　宋寿柏

(安徽师范大学数学系,　芜湖 241000)

摘　要

本文研究近严格凸与最佳逼近的关系. 证明了Banach 空间 X 是近严格凸的当且仅当 X 的每个子空间是紧-半-切比晓夫空间.

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd.　All rights reserved.