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On the Structure of a Radical of Lattice-Ordered Rings*

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Abstract We describe the P- radical and the l-B radical of an l-ring from a different angle and obtain the structure theorem of l-Q semisimple rings. Furthermore, l-Q radical rings are discussed. Lastly, we consider the l-Q radical $Q(R_n)$ of a full l-matrix l-ring R_n over an l-ring R. We show that $Q(R_n)=(Q(R))_n$.

Keywords l-Q ideal, l- Q radical, l-Q radical ring, l-Q semisimple ring.

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1. Introduction

In this paper we discuss the P-radical of a lattice- ordered ring (l-ring) anew from a different angle using the concept of l-quasi nilpotent ideal (l-Q ideal), and obtain the structure and some properties of the P-radical.

We first collect some basic definitions and properties of l-rings in [2- 5]. Throughout this paper R will denote an l-ring if not specified. $R^+=\{a\in R: a\geq 0\}$. Let $\langle S\rangle$ be an l-ideal of R generated by the subset S of R. We use the term l-prime l-ideal instead of prime l-ideal given in [4, Definition 2.1]. An l-ideal P of R is l-prime if $I\subseteq P$ or $J\subseteq P$ whenever I and J are l-ideals of R with $IJ\subseteq P$; a nonzero l-ring R is l-prime if $\{0\}$ is an l-prime l-ideal; the P- radical of R is the intersection of all l-prime l-ideals of R (see [4, Definition 2.1 and 2.8]). The product of two (left, right, two- sided) l-ideals A, B of R is the (left, right, two- sided) l-ideal $\langle AB\rangle$ of R, where $\langle AB\rangle=\{c\in R: |c|\leq ab;\ a\in A,\ b\in B\}$ [5, p169]. An l-homomorphism (l-isomorphism) between two l-rings is a mapping which is a ring and lattice homomorphism (isomorphism).

Proposition 1.1 (1) *Let an l-ring R^* be an l-homomorphic image of R ($R\overset{\varphi}{\sim}R^*$), A an l-ideal of R, then $R^*\cong R/\ker\varphi$ and $\varphi(A)=(A+\ker\varphi)/\ker\varphi$. Moreover, if $A\supseteq\ker\varphi$, then $R/A\cong R^*/\varphi(A)$.*

(2) *Let S be an l-subring of R, A an l-ideal of R, then $(S+A)/A\cong S/(S\cap A)$.*

(3) *Let M be an l-ideal of R, then any l-ideal of R/M has the form A/M where A is an l-ideal of R containing M, and $R/A\cong(R/M)/(A/M)$.*

(4) *Let A be a left (right) l-ideal of R, and T an l-ideal of A, then $\langle AT)=\{c\in R: |c|\leq at,\ a\in A,\ t\in T\}$ ($(TA\rangle=\{c\in R: |c|\leq ta, a\in A, t\in T\}$) is a left (right) l-ideal of R.*

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2 l-Q ideals and the l-Q radical of an l-ring

In this section some important properties of l-Q ideals, the l-Q radical of an l-ring are discussed. It is shown that the l-Q radical coincides with the l-B radical.

Definition 2.1 *A left (right, or two-sided) l-ideal A of R is called a left (right, or two-sided) l-Q ideal of R, if for any l-ideal M of R, $A^* = (A + M)/M \neq \{0^*\}$ implies that A^* contains a nonzero nilpotent left (right, or two-sided) l-ideal of $R^* = R/M$.*

An l-Q ideal may not be nilpotent, [5, Example 3.19] forms a case in point.

Proposition 2.1 (1) *Any nonzero l-Q ideal A of R contains a nonzero nilpotent l-ideal of R.*

(2) *A nilpotent l-ideal is an l-Q ideal.*

(3) *Let A be an l-Q ideal of R, M an l-ideal of R, then $(A + M)/M$ is an l-Q ideal of R/M.*

Similar results can be stated for left (respectively right) l-ideals.

Proposition 2.2 (1) *A left (right) l-Q ideal A of R can be embedded in an l-Q ideal $A + \langle AR \rangle$ $(A + \langle RA \rangle)$ of R.*

(2) *Suppose A and B/A are l-Q ideals of R and R/A respectively, then B is an l-Q ideal of R.*

(3) *Any sum of l-Q ideals of R is an l-Q ideal of R. In particular, the sum of all l-Q ideals of R is an l-Q ideal of R.*

Since the l-radical $N(R)$ of R is the union of all the nilpotent l-ideals of R ([3, Th. 5]), l-radical is an l-Q ideal by Proposition 2.1 (2) and Proposition 2.2 (3). However, $N(R)$ may not be nilpotent, as is shown by [5, Example 3.19]. This example also shows that nilpotent and l-quasi nilpotent are different in general.

Definition 2.2 *The sum $Q(R)$ of all l-Q ideals of R is called the l-Q radical of R. If $Q(R) = R$, then R is called an l-Q radical ring; if $Q(R) = \{0\}$, then R is called an l-Q semisimple ring.*

Theorem 2.1 (1) *The l-Q radical of R is the largest l-Q ideal of R, and contains all left (and right) l-Q ideals and left (and right) nilpotent l-ideals of R.*

(2) *$R/Q(R)$ is an l-Q semisimple ring.*

(3) *R is an l-Q radical ring if and only if every nonzero l-homomorphic image of R contains a nonzero nilpotent l-ideal.*

Proof It follows immediately from Proposition 2.2 and 2.1.

Definition 2.3 *We define in R an l-ideal $N(v)$ for every ordinal number v as follows*

(*i*) $N(0) = \{0\}$.

Let us assume that $N(v)$ is already defined for every $v < u$.

(*ii*) *If $u = v + 1$ is not a limit ordinal number, $N(u)/N(v)$ is the sum of all nilpotent l-ideals of $R/N(v)$.*

(*iii*) *If u is a limit ordinal number, $N(u) = \sum_{v<u} N(v)$.*

Since every l-ring is a set, for every l-ring there exists an ordinal number with $N(w) = N(w+1)$. We denote this l-ideal $N(w)$ of R by $B(R)$, which is called the Baer radical (also l-B radical) of R. $B(R)$ is characterized by the fact that $R/B(R)$ has no nonzero nilpotent l-ideals and $B(R)$ is the smallest l-ideal in our chain that gives such a factor l-ring.

If $B(R) = \{0\}$, then we say that R is an l-B semisimple ring; if $B(R) = R$, then we say that R is an l-B radical ring.

Part (2) of the next Proposition follows from Proposition 2.1 (1) and (2).

Proposition 2.3 (1) *The l-B radical $B(R)$ of R is a nil l-ideal of R, and $R/B(R)$ is an l-B semisimple ring.*

(2) *R is an l-B semisimple ring if and only if R is an l-Q semisimple ring.*

Theorem 2.2 *The l-Q radical $Q(R)$ coincides with the l-B radical $B(R)$ of R, this implies that R is an l-Q radical ring if and only if R is an l-B radical ring.*

Proof It is easy to prove that $Q(R) = B(R)$ via transfinite induction.

From [5, Example 3.19] we already know that the union $N(R)$ of all the nilpotent l-ideals of R may not be nilpotent, although it must be nil. Furthermore, [3, p46, Example 8] shows that $R/N(R)$ may have nonzero nilpotent l-ideals. It also shows that the l-Q radical and the l-radical of R are in general, different.

For d-rings a much stronger statement can be made.

An f-ring is an l-ring in which

$$a \wedge b = 0 \text{ and } c \geq 0 \text{ imply } ca \wedge b = ac \wedge b = 0.$$

An important identity satisfied by any f-ring is $|xy| = |x||y|$ ([3, p57, Corollary 1]). An l-ring satisfying this identity is usually called a distributive l-ring or d-ring. So any f-ring is a d-ring, but there is a d-ring which is not an f-ring ([3, p59]).

Theorem 2.3 *If R is a d-ring, then the l-Q radical $Q(R)$ of R is the set of all nilpotent elements of R.*

Proof Let $\overline{R} = R/Q(R)$. Clearly $\overline{R}$ is a d-ring. Suppose that $\overline{a} \in \overline{R}^+$ and $\overline{a}\overline{R} = \overline{0}$. Then $\overline{a} \in N(\overline{R}) \subseteq Q(\overline{R}) = \{\overline{0}\}$ by the definition of l-radical $N(R)$ [3, p45, Definition] and Theorem 2.1 (2). Hence $\overline{R}$ is an f-ring by [3, p58-59, Lemma 1 and Theorem 14]. Suppose that $x \in R$ and $x^n = 0$. Then $\overline{x^n} = \overline{0}$ and $\overline{x} \in N(\overline{R}) = Q(\overline{R}) = \{\overline{0}\}$ by [3, p63, Theorem 16]. Thus $x \in Q(R)$.

Corollary *If R is a d-ring, then $R/Q(R)$ is an f-ring.*

3 The P-radical of R and l-Q semisimple rings

In this section we give a comprehensive characterization of the P-radical introduced in [4] and show the structure theorem of l-Q semisimple rings. In order to do this we need the following.

Definition 3.1 *R is called an l-semiprime l-ring, if R contains no nonzero nilpotent l-ideal; an l-ideal A of R is called an l-semiprime l-ideal, if R/A is an l-semiprime l-ring.*

A useful proposition follows directly from Proposition 2.3 and this definition.

Proposition 3.1 *The following statements are equivalent.*

(1) *R is an l-Q semisimple ring.*

(2) *R is an l-B semisimple ring.*

(3) *R is an l-semiprime l-ring.*

We may characterize $Q(R)$ as follows

Theorem 3.1 *The l-Q radical $Q(R)$ of R coincides with the intersection of all l-semiprime l-ideals $A_\alpha$lpha of R.*

Proof Let $N = \cap \{A_\alpha lpha: A_\alpha lpha$ is an l-semiprime l-ideal of $R\}$. By Theorem 2.1 (2) and Proposition 3.1, $Q(R)$ is an l-semiprime l-ideal, thus $N \subseteq Q(R)$. Assume B/N is a nilpotent l-ideal of R/N, since $(B + A_\alpha)/A_\alpha \cong B/(A_\alpha \cap B) \cong (B/N)/((A_\alpha \cap B)/N)$, $(B + A_\alpha)/A_\alpha$ is a nilpotent l-ideal, so $B \subseteq A_\alpha$ and $B \subseteq N$. Applying Proposition 3.1, R/N is an l-Q semisimple ring. But $Q(R)/N$ is an l-Q ideal of R/N by Theorem 2.1 (1) and Proposition 2.1 (3). Whence $Q(R) \subseteq N$. Thus $Q(R) = N$.

This theorem characterizes the l-Q radical of R and will be used in the rest of the paper. Moreover, we have

Theorem 3.2 *The following subsets of R coincide with the P-radical $P(R)$ of R:*

(1) *the l-Q radical $Q(R)$ of R,*

(2) *the l-B radical $B(R)$ of R,*

(3) *the intersection of all l-semiprime l-ideals of R.*

Proof It is clear that an l-ideal A of R is l-semiprime if and only if $N(R/A)$ is zero. Using Theorem 2.2 and 3.1, and [4, 2.12], we have $Q(R) = B(R) = \cap \{A_\alpha: A_\alpha$ is an l-semiprime l-ideal of $R\} = \cap \{A_\alpha: A_\alpha$ is an l-ideal of R and $N(R/A_\alpha)$ is zero$\} = P(R)$.

Theorem 3.3 *R is an l-Q semisimple ring if and only if it is a subdirect sum of some l-prime l-rings.*

Theorem 3.3 follows directly from Proposition 3.1 and [1, Corollary 8.5.8].

4 l-Q radical rings

In this section we discuss l-Q radical rings in order to obtain the structure of l-Q ideals. The next theorem is a basic result.

Theorem 4.1 *Every l-homomorphic image R^* and every l-subring M of an l-Q radical ring R are l-Q radical rings. In particular, l-ideals of an l-Q radical ring are also l-Q radical rings.*

Proof It is easy to see from Theorem 2.1 (3) that R^* is also an l-Q radical ring.

It remains to prove $M = B(M)$ by Theorem 3.2 and Definition 2.2. Let $N(v)$ be as in Definition 2.3, S the complementary set of $B(M)$ in M. Clearly $S \cap N(0) = \varnothing$. Now assume $S \cap N(v) = \varnothing$ for each ordinal number $v < u$. If u is a limit ordinal number, then $S \cap N(u) = \varnothing$ by the transfinite inductive hypothesis; if $u = v + 1$ is not a limit ordinal number, then $N(u)$ is the sum of all l-ideals K_α of R such that K_α contains $N(v)$ and $K_\alpha/N(v)$ is nilpotent. Assume $M_1 = S \cap N(u)$ is a nonempty set, then there is $a \in M_1$. Let $\langle a\rangle_M$ and $\langle a\rangle$ be l-ideals generated by $\{a\}$ in M and in R respectively. Since $a \in M_1$, there exist α and a positive integer n such that $\langle a\rangle^n \subseteq K_\alpha^n \subseteq N(v)$, hence $\langle a\rangle_M^n \subseteq M \cap N(v)$ from $\langle a\rangle_M \subseteq \langle a\rangle$. But since $S \cap N(v) = \varnothing$ by the transfinite inductive hypothesis, then $\langle a\rangle_M^n \subseteq M \cap N(v) = B(M) \cap N(v) \subseteq B(M)$. Applying Theorem 3.2 we obtain $\langle a\rangle_M \subseteq B(M)$ and $a \in B(M) \cap S$, which contradicts the choice of S, thus $M_1 = N(u) \cap S = \varnothing$. Whence $S \cap B(R) = \varnothing$ via transfinite induction. Since $R = B(R)$, $S = S \cap M \subseteq S \cap R = S \cap B(R) = \varnothing$. Therefore $M = B(M)$.

Definition 4.1 *Let X be a nonempty subset of R, the left l-annihilator of X in R is $l(X) = \{r \in R: |r|x = 0$ for each $x \in X\}$.*

A similar definition can be made for right l-annihilator.

Proposition 4.1 (1) *Suppose that X is a nonempty subset of R, if $|x| \in X$ for each $x \in X$, then the left (right) l-annihilator of X in R is a left (right) l-ideal of R.*

(2) *The left (right) l-annihilator of a left (right) l-ideal of R in R is an l-ideal of R.*

Theorem 4.2 (1) *Any l-ideal A of an l-Q semisimple ring R is also an l-Q semisimple ring.*

(2) *If A is an l-ideal of R contained in the l-Q radical $Q(R)$ of R, then the l-Q radical of R/A is $Q(R)/A$.*

Proof It is immediate from Proposition 3.1 and 4.1, and Theorem 3.1 and 2.1.

Next we discuss the structure of l-Q radical rings and l-Q ideals

Theorem 4.3 *A left (right, two-sided) l-ideal A of R is a left (right, two-sided) l-Q ideal if and only if A is an l-Q radical ring.*

Proof Sufficiency. Without loss of generality we now assume only that A is a nonzero left l-ideal. Let M be any l-ideal of R, and A be not contained in M. Then $A^* = (A + M)/M$ is a nonzero left l-ideal of $R^* = R/M$. Using the hypothesis that A is an l-Q radical ring, Theorem 4.1 and $A/(A \cap M) \cong (A + M)/M = A^*$, we have that A^* is an l-Q radical ring. Assume that B^* is the right l-annihilator of A^* in A^*, then $B^{*2} \subseteq A^*B^* = \{0^*\}$, clearly B^* is a nilpotent l-ideal of A^* by Proposition 4.1. If $B^* = A^*$, then A is a left l-Q ideal of R; if $B^* \neq A^*$, from Theorem 2.1 (3) we may assume that L^{**} is a nonzero nilpotent l-ideal of $A^{**} = A^*/B^*$. Let φ be the natural l-homomorphism of A^* onto A^{**}, and $L^* = \varphi^{-1}(L^{**})$ the inverse l-homomorphic image of L^{**} under φ, then $L^{**} = L^*/B^*$, L^* is nilpotent, and $A^*L^* \neq \{0^*\}$. Hence by Proposition 1.1 (4) the l-ideal $\langle A^*L^*\rangle$ is a nonzero nilpotent left l-ideal of R^*. It follows from Definition 2.1 that A is a left l-Q ideal of R.

Necessity. Assume first that A is a two-sided l-Q ideal. Let I be an l-ideal of A with $I \neq A$. We will show that the l-ring A/I contains a nonzero nilpotent l-ideal. Let $Q = \langle\cup$ (all l-ideals of R that are contained in I)$\rangle$. It is easily seen that Q is an l-ideal of R and that $Q \subseteq I$. Since A is an l-Q ideal, there is an l-ideal L of R such that $Q \subseteq L \subseteq A + Q = A$, $L/Q \subseteq (A +$

$Q)/Q$ and L/Q is a nonzero nilpotent l-ideal of R/Q. Now $L \nsubseteq I$, since otherwise, we would have $L \subseteq Q$ and that L/Q is the zero l-ideal in R/Q. So $(L+I)/I$ is a nonzero l-ideal in A/I. Since L/Q is nilpotent in R/Q, there is a positive integer n such that $(L/Q)^n = 0$ in R/Q. So $L^n \subseteq Q \subseteq I$. This implies $(L+I)/I$ is nilpotent in A/I. Since I was chosen arbitrarily, we have shown that any l-homomorphic image of A contains a nonzero nilpotent l-ideal. Thus by Theorem 2.1 (3), A is an l-Q radical ring.

Next assume that A is a left (right) l-Q ideal. Then by Proposition 2.2, A can be embedded in the l-Q ideal $A + \langle AR \rangle$ $(A + \langle RA \rangle)$. By our previous discussions, $A + \langle AR \rangle$ $(A + \langle RA \rangle)$ is an l-Q radical ring. Now A is an l-subring of $A + \langle AR \rangle$ $(A + \langle RA \rangle)$. By Theorem 4.1, A is an l-Q radical ring.

Theorem 4.4 (1) *The l-Q radical $Q(R)$ of R and every l-subring of $Q(R)$ are also l-Q radical rings.*

(2) *Every left (right, or two-sided) l-ideal A of R is a left (right, or two-sided) l-Q ideal of R if and only if A is contained in $Q(R)$.*

(3) *If A is an l-ideal of R, then $Q(A) = Q(R) \cap A$.*

Proof The proof of Part (1) is immediate from Theorem 2.1 (1), Theorems 4.3 and 4.1.

(2) Necessity. It follows immediately from Theorem 2.1 (1).

Sufficiency. Since $A \subseteq Q(R)$, by Part (1), we know that A is an l-Q radical ring. Hence by Theorem 4.3 A is a left (right, or two-sided) l-Q ideal of R.

(3) Using part (1), Theorem 4.3, and part (2), we have $Q(R) \cap A \subseteq Q(A)$. On the other hand, by Theorem 2.1 (2) and 4.2 (1), we have that $A/(A \cap Q(R)) \cong (A + Q(R))/Q(R)$ is an l-Q semisimple ring. It follows from Proposition 3.1 and Theorem 3.1 that $Q(A) \subseteq A \cap Q(R)$. Hence (3) holds.

Corollary *If M is an l-subring of R, then $Q(M) \supseteq Q(R) \cap M$. (namely $B(M) \supseteq B(R) \cap M$) }*

Theorem 4.5 *If an l-ring R^* is an l-homomorphic image of R $(R \overset{\varphi}{\sim} R^*)$ and $K = \ker\varphi \subseteq Q(R)$, then an l-ideal $A \supseteq K$ of R is the l-Q radical of R if and only if $\varphi(A)$ is the l-Q radical of R^*.*

Proof Immediate.

5. l-Q radical for full matrix lattice-ordered rings

The purpose of this section is to discuss the l-Q radical of full matrix lattice-ordered rings (full l-matrix l-rings).

Theorem 5.1 *An l-ring R without identity can be embedded naturally in an l-ring R_0 with identity such that*

(1) *Every left (right, two-sided) l-ideal A of R is also a left (right, two-sided) l-ideal of R_0.*

(2) *A is a nil (nilpotent) l-ideal of R if and only if A is a nil (nilpotent) l-ideal of R_0.*

(3) *If A is an l-Q ideal of R, then A is also an l-Q ideal of R_0.*

(4) *The l-Q radical $Q(R)$ of R coincides with the l-Q radical $Q(R_0)$ of R_0.*

Proof Let $R_0 = \{a + ne \mid a \in R,\ n$ is an integer$\}$, then R_0 can be made to be an l-ring with identity e in which R is an l-subring, by defining

i $a + ne = b + me$ if and only if $a = b$, $m = n$;
$a + ne \in R$ if and only if $n = 0$.
ii $(a + ne) + (b + me) = (a + b) + (n + m)e$.
iii $(a + ne)(b + me) = ab + nb + ma + nme$.
iv $a + ne \leq b + me$ if and only if $a \leq b$, $n \leq m$.

There is no loss of generality in assuming that A is a left l-ideal of R. For all $x = a + ne \in R_0$, $b \in A$ and $|a + ne| \leq |b|$, then we have $(a + ne) \vee (-a - ne) = |a| + \max\{n, -n\}e \leq |b|$. It follows that $|a| \leq |b|$ and $n = 0$, that is $x = a \in A$. Hence (1) holds by iii

(2) The necessity is immediate from (1). The converse is trivial since for any $x = a + ne \in A$, there exists a positive integer k such that $(a + ne)^k = 0$, by i- iv, we have $n = 0$. It follows that $A \subseteq R$. This completes the proof of (2).

(3) Since A is an l-Q ideal of R, A is an l-ideal of R_0 by (1). If M_0 is an l-ideal of R_0 with $A \nsubseteq M_0$, then $M = M_0 \cap R \nsupseteq A$, and $(A + M)/M$ contains a nonzero nilpotent l-ideal B/M of R/M. It follows that B sup$setM$ and $B \neq M$, $B \nsubseteq M_0$ and $(B + M_0)/M_0 \cong B/(B \cap M_0) = B/M$ is a nonzero nilpotent l-ideal of R_0/M_0. Since $(A + M_0)/M_0 \supseteq (B + M_0)/M_0$, A is an l-Q ideal of R_0.

(4) It is clear from Theorem 2.1 (1) and Part (3) that $Q(R) \subseteq Q(R_0)$. Conversely, suppose that $B_0/Q(R)$ is a nilpotent l-ideal of $R_0/Q(R)$, then $(B_0 \cap R)/Q(R)$ is a nilpotent l-ideal of $R/Q(R)$. Therefore $B_0 \cap R = Q(R)$ by Theorem 2.1 (2), Proposition 3.1 and Definition 3.1. Since $B_0/Q(R)$ is nilpotent, it follows from i- iv that $B_0 \subseteq R$. Thus $R_0/Q(R)$ is an l-semiprime l-ring. By Theorem 3.1 this implies $Q(R_0) \subseteq Q(R)$. Hence (4) holds

Let R_n be the full matrix ring of all $n \times n$ matrices $Y = (y_{ij})$ over an l-ring R. Then, by defining $Y \leq Z$ to mean that $y_{ij} \leq z_{ij}$ for all $i, j = 1, 2, \cdots, n$, we get an l-ring, which is called a full l-matrix l-ring over R.

Suppose η is an l-homomorphism of an l-ring R into an l-ring S. Then η_n: $R_n \to S_n$ is the ring homomorphism induced by η on the full l-matrix l-ring. That is $\eta_n(y_{ij}) = (\eta(y_{ij}))$.

Lemma 5.1 *Let η be a surjective l-homomorphism of R onto S, and let $\ker\eta = A$. Then η_n is a surjective l-homomorphism of R_n onto S_n, and $\ker\eta_n = A_n$. That is $R_n/A_n \cong S_n \cong (R/A)_n$.*

Theorem 5.2 *Let R be an l-ring with identity e, A an l-ideal of R, and let A_n denote the set of $n \times n$ matrices with entrices in A. Then*

(1) *The map $A \to A_n$ is a bijective map of the set of l-ideals in R onto the set of l-ideals in R_n.*

(2) A is a nilpotent l-ideal of R if and only if A_n is a nilpotent l-ideal of R_n.

(3) A is an l-Q ideal of R if and only if A_n is an l-Q ideal of R_n.

(4) A is an l-prime l-ideal of R if and only if A_n is an l-prime l-ideal of R_n.

(5) A is an l-semiprime l-ideal of R if and only if A_n is an l-semiprime l-ideal of R_n.

Proof (1) It is evident that if A is an l-ideal of R, then A_n is an l-ideal of R_n. Conversely, assume that K is an l-ideal of R_n. Then, by [6, p471, Proposition 7], there exists a ring ideal N of R such that $K = N_n$. For any $x \in R$, $a \in N$, and $|x| \le |a|$, we have $aE \in N_n$ and $|xE| \le |aE|$, where E is the unit matrix, or identity, of R_n. Since $K = N_n$ is an l-ideal of R_n, there is $xE \in N_n$. This implies $x \in N$. It follows that N is an l-ideal of R.

(2) If A is a nilpotent l-ideal, then there exists a positive integer k such that $A^k = \{0\}$. Hence $(A_n)^k = \{(0)\}$, where (0) is the $n \times n$ matrix whose entries are all 0. Conversely, if A_n is a nilpotent l-ideal of R_n, that is $(A_n)^k = \{(0)\}$, then for any $a_1, a_2, \cdots, a_k \in A$, $(0) = (a_1E)(a_2E)\cdots(a_kE) = (a_1a_2\cdots a_k)E$. It follows that $A^k = \{0\}$.

(3) Suppose that A_n is an l-Q ideal of R_n. Take any l-ideal M of R with $A \nsubseteq M$, then $A_n \nsubseteq M_n$ by (1). Hence $(A_n + M_n)/M_n$ contains a nonzero nilpotent l-ideal B_n/M_n of R_n/M_n. Since $(B/M)_n \cong B_n/M_n \subseteq (A_n + M_n)/M_n = (A + M)_n/M_n \cong ((A + M)/M)_n$ by Lemma 5.1, we obtain that $B/M \subseteq (A + M)/M$ is nilpotent. Thus A is an l-Q ideal of R. Similarly, the converse holds.

(4) We first show the following fact:

$$\langle I_nJ_n\rangle = \langle (IJ)_n\rangle = \langle IJ\rangle_n \qquad (*)$$

for every pair of l-ideals I, J of R. Since $I_nJ_n \subseteq (IJ)_n \subseteq \langle IJ\rangle_n$, $\langle I_nJ_n\rangle \subseteq \langle (IJ)_n\rangle \subseteq \langle\langle IJ\rangle_n\rangle = \langle IJ\rangle_n$. Conversely, for every $Y = (y_{ij}) \in \langle IJ\rangle_n$, there are $y_{ij} \in \langle IJ\rangle$, $i, j = 1, 2, \cdots, n$. By [5, p169] there exist $a_{ij} \in I$, $b_{ij} \in J$ such that $|y_{ij}| \le a_{ij}b_{ij}$, hence $|Y| = (|y_{ij}|) \le (a_{ij}b_{ij}) = \sum_{\{i,j\}}(a_{ij}E_{ij})(b_{ij}E_{ij}) \in I_nJ_n$, where E_{ij} is the $n \times n$ matrix with (i, j)-entry e and all other entries 0. This implies $Y \in \langle I_nJ_n\rangle$. Thus $(*)$ holds. From (1), [4, pp. 71-72] and $(*)$ we can easily obtain that A is an l-prime l-ideal of R if and only if A_n is an l-prime l-ideal of R_n.

(5) It is immediate from Part (1) and (2) and Lemma 5.1.

Corollary *If R is an l-ring with identity, then the l-Q radical of R_n is the set of all $n \times n$ matrices with entries in the l-Q radical of R. That is $Q(R_n) = (Q(R))_n$.*

Theorem 5.3 *For any l-ring R, the l-Q radical $Q(R_n)$ of R_n is the full l-matrix l-ring $(Q(R))_n$ over the l-Q radical $Q(R)$ of R. That is $Q(R_n) = (Q(R))_n$.*

Proof It is suffices to consider the case R without identity, by Corollary of Theorem 5.2. From Theorem 5.1, R can be embedded in an l-ring $\overline{R}$ with identity in which R is an l-ideal of $\overline{R}$ and $Q(R) = Q(\overline{R})$. Then, by Theorem 5.2 and its Corollary, R_n is an l-ideal of $\overline{R}_n$ and

$$Q(\overline{R}_n) = (Q(\overline{R}))_n = (Q(R))_n \subseteq R_n.$$

By Theorem 4.4 (1) and (3) we obtain $Q(\overline{R}_n) = Q(Q(\overline{R}_n)) \subseteq Q(R_n)$; By Theorem 4.4 (3) we get $Q(R_n) \subseteq Q(\overline{R}_n)$. Whence $Q(R_n) = Q(\overline{R}_n) = (Q(R))_n$.

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格序环的一个根的结构

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摘　　要

从不同角度刻画了格序环 R 的 P- 根和 l-B 根，并对 l-Q 根环进行了讨论. 揭示了 R 及 R 上的全矩阵环 R_n 的 l-Q 根，l-Q 理想，素 l-理想，半素 l-理想之间的关系