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单纯形上 Stancu 算子对光滑函数的逼近误差的渐近展开*

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摘 要 本文定义了 m 维空间内一般单纯形上的 Stancu 算子, 并给出了它对光滑函数的点态逼近误差的高阶渐近公式

关键词 单纯形, Stancu 算子, $C^{2N}(\sigma)$ 空间, 渐近展开.
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1 引 言

Stancu 算子是 Bernstein 算子的一种推广, 首先在[1]内提出并研究 近年来, 对高维单纯形上 Bernstein 算子的研究有很大进展[3], [4], [5]. 特别地, [6], [7]中建立了高维单纯形上 Bernstein 算子对高阶光滑函数的点态逼近误差的渐近展开式, 其结果大大推广并深化了经典的 Voronovskaja 结果 本文目的是定义 m 维($m\geq 2$) 空间内一般单纯形上的 Stancu 算子, 并把[6], [7]的结果拓广到该算子上去

设 $v^{(i)}\in R^m$, $i=0,1,\cdots,m$ 是仿射意义下相互独立的点, 即$\{v^{(i)}-v^{(0)}\}_{i=1}^m$ 线性无关 由此 $m+1$ 个点做成的最小凸包称为 m 维空间中的一般单纯形, 记作 $\sigma=[v^{(0)},v^{(1)},\cdots,v^{(m)}]$, $\forall x=(x_1,\cdots,x_m)\in\sigma$, 它关于 σ 的重心坐标 $\lambda=(\lambda_0,\lambda_1,\cdots,\lambda_m)\in R^{m+1}$, 这里有 $x=\sum_{j=0}^m\lambda_j v^{(j)}$, $\sum_{j=0}^m\lambda_j=1$. 记 $B_\alpha(\lambda)=\binom{|\alpha|}{\alpha}=\frac{(\alpha_0+\cdots+\alpha_m)!}{\alpha_0!\cdots\alpha_m!}\lambda_0^{\alpha_0}\cdots\lambda_m^{\alpha_m}$, $\alpha=(\alpha_0,\cdots,\alpha_m)\in Z_+^{m+1}$ ($m+1$ 维非负整向量空间). 对于 α, 记 $x_\alpha^{(i)}=\frac{1}{n}\sum_{j=0}^m(\alpha_j+sq_{ij})v^{(j)}$, $q_{ij}=\begin{cases}0 & i\neq j\\ 1 & i=j\end{cases}$. $\forall f(x)\in c(\sigma)$, 定义

$$M_{ns}^{(i)}(f;x)=\sum_{|\alpha|=n-s}f(x_\alpha^{(i)})B_\alpha(\lambda).$$

从而单纯形 σ 上的 Stancu 算子定义为 $M_{ns}(f;x)=\lambda\overline{M}_{ns}=\sum_{i=0}^m\lambda_i M_{ns}^{(i)}(f;x)$. $s\in N$.

不难验证当参数 $s=0$ 或 1 时, $M_{ns}(f;x)$ 正好是 σ 上的 Bernstein 算子[6, 7]. 本文的主要结果放在 3, 而需要的引理在 2

2 引 理

记 $S_\gamma^{(i)}(x)=n^{|\gamma|}\sum_{|\alpha|=n-s}(x_\alpha-x)^\gamma B_\alpha(\lambda)$, $\gamma\in Z_+^m$, 则有

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引理 1

$$S^{(i)}_{\gamma+e^k}(x) - s(v_k^{(i)} - x_k)S^{(i)}_{\gamma}(x) = \sum_{j=0}^{m}\lambda_j\sum_{\substack{\beta\le\gamma\\ \beta\ne\gamma}}[n(v_k^{(j)} - x_k)s^{(i)}_{\beta}(x) + S(v_k^{(i)} - v_k^{(j)})S^{(i)}_{\beta}(x)$$
$$- S^{(i)}_{\beta+e^k}(x)]\binom{\gamma}{\beta}(v^{(j)} - v^{(0)})^{\gamma-\beta}\quad(|\gamma|\ge 1)$$

其中 $e^k\in R^m$ **是第** k **个分量为** 1 **的单位向量,** $\binom{\gamma}{\beta} = \binom{\gamma_1}{\beta_1}\cdots\binom{\gamma_m}{\beta_m}$.

证明 **记** $g^{(i)}(t;x) = \sum_{\gamma\in Z_+^m}\frac{S^{(i)}_{\gamma}(x)}{\gamma!}(\frac{t}{n})^{\gamma}$, $x, t\in R^m$, **则**

$$g^{(i)}(t;x) = \sum_{|\alpha|=n-s}e^{(x_\alpha^{(i)}-x)\cdot t}B_\alpha(\lambda) = e^{(\frac{s}{n}v^{(i)}-x)\cdot t}(\sum_{j=0}^{m}\lambda_j e^{\frac{v^{(j)}}{n}\cdot t})^{n-s},$$

所以有

$$\frac{\partial g^{(i)}}{\partial t_k} = \sum_{j=0}^{m}\lambda_j(1 - e^{(\frac{v^{(j)}}{n}-\frac{v^{(0)}}{n})\cdot t})[\frac{\partial g^{(i)}}{\partial t_k} + (x_k - v_k^{(j)})g^{(i)} - \frac{s}{n}(v_k^{(i)} - v_k^{(j)})g^{(i)}]$$
$$- \frac{s}{n}(x_k - v_k^{(i)})g^{(i)}.$$

注意到

$$\frac{\partial g^{(i)}}{\partial t_k} = \frac{1}{n}\sum_{\gamma\in Z_+^m}\frac{S^{(i)}_{\gamma+e^k}(x)}{\gamma!}(\frac{t}{n})^{\gamma},\tag{1}$$

又

$$1 - e^{(\frac{v^{(j)}}{n}-\frac{v^{(0)}}{n})\cdot t} = -\sum_{\substack{\alpha\in Z_+^m\\ \alpha\ne 0}}\frac{(v^{(j)} - v^{(0)})}{\alpha!}(\frac{t}{n})^{\alpha},$$

从而有

$$\frac{\partial g^{(i)}}{\partial t_k} = \sum_{\gamma\in Z_+^m}(\sum_{j=0}^{m}\lambda_j\sum_{\beta\le\gamma}[(v_k^{(j)} - x_k)S^{(i)}_{\beta}(x) + \frac{s}{n}(v_k^{(j)} - v_k^{(i)})S^{(i)}_{\beta}(x)$$
$$- \frac{1}{n}S^{(i)}_{\beta+e^k}(x)]\binom{\gamma}{\beta}(v^{(j)} - v^{(0)})^{\gamma-\beta} + \frac{s}{n}(x_k - v_k^{(i)})S^{(i)}_{\gamma}(x))\frac{1}{\gamma!}(\frac{t}{n})^{\gamma},\tag{2}$$

比较(1) **和**(2) **中** $(\frac{t}{n})^{\gamma}$ **前的系数便得结论 证毕**

引理 2

$$S^{(i)}_{\gamma+e^p}(x) = \sum_{k,l=0}^{m}\lambda_k\lambda_l v_p^{(k)}[(D_{\lambda_k} - D_{\lambda_l})S^{(i)}_{\gamma}(x)$$
$$+ n\sum_{j=1}^{m}\gamma_j(v_j^{(k)} - v_j^{(l)})S^{(i)}_{\gamma-e^j}(x)] + S(v_p^{(i)} - x_p)S^{(i)}_{\gamma}(x),$$

其中 $D_{\lambda_k} = \frac{\partial}{\partial\lambda_k}, p = 1, 2, \cdots, m\ (|\gamma|\ge 1)$

证明 **经直接计算得**

$$(D_{\lambda_k} - D_{\lambda_l})S^{(i)}_{\gamma}(x) = -n^{|\gamma|}\sum_{|\alpha|=n-s}\sum_{j=1}^{m}\gamma_j(x_\alpha^{(i)} - x)^{\gamma-e^j}B_\alpha(\lambda)(\frac{\partial x_j}{\partial\lambda_k} - \frac{\partial x_j}{\partial\lambda_l})$$

$$+\sum_{|\alpha|=n-s}(x_\alpha^{(i)}-x)^{\gamma}\left(\frac{\alpha_k}{\lambda_k}-\frac{\alpha_l}{\lambda_l}\right)B_\alpha(\lambda),$$

由 $x=\sum_{j=0}^{m}\lambda_j v^{(j)}$ 知 $\frac{\partial\alpha_j}{\partial\lambda_k}-\frac{\partial\alpha_j}{\partial\lambda_l}=v_j^{(k)}-v_j^{(l)}$, 从而

$$\lambda_k\lambda_l\left[(D_{\lambda_k}-D_{\lambda_l})S_\gamma^{(i)}(x)+n\sum_{j=1}^{m}\gamma_j(v_j^{(k)}-v_j^{(l)})S_{\gamma-e^j}(x)\right]$$

$$=\lambda_l\left[n^{|r|}\sum_{|\alpha|=n-s}B_\alpha(\lambda)(x_\alpha^{(i)}-x)^{\gamma}(\alpha_k+sq_{ik}-n\lambda_k)\right]$$

$$-\lambda_k\left[n^{|r|}\sum_{|\alpha|=n-s}B_\alpha(\lambda)(x_\alpha^{(i)}-x)^{\gamma}(\alpha_l+sq_{il}-n\lambda_l)\right]$$

$$+n^{|r|}\sum_{|\alpha|=n-s}B_\alpha(\lambda)(x_\alpha^{(i)}-x)^{\gamma}(sq_{il}\lambda_k-sq_{ik}\lambda_l),$$

两边同乘 $v_p^{(k)}$, 并对 k 从 0 到 m 求和, 注意到

$$\sum_{k=0}^{m}(\alpha_k+sq_{ik}-n\lambda_k)v_p^{(k)}=n(x_\alpha^{(i)}-x)_p,$$

得

$$\sum_{k=0}^{m}\lambda_k\lambda_l v_p^{(k)}\left[(D_{\lambda_k}-D_{\lambda_l})S_\gamma^{(i)}(x)+n\sum_{j=1}^{m}\gamma_j(v_j^{(k)}-v^{(l)})S_{\gamma-e^j}^{(i)}(x)\right]$$

$$=\lambda_l S_{\gamma+e^p}^{(i)}(x)-x_p n^{|\gamma|}\sum_{|\alpha|=n-s}(x_\alpha^{(i)}-x)^{\gamma}B_\alpha(\lambda)(\alpha_l+sq_{il}-n\lambda_l)$$

$$+S_\gamma^{(i)}(x)\sum_{k=0}^{m}(sq_{il}\lambda_k-sq_{ik}\lambda_l)v_p^{(k)},$$

两边对 l 从 0 到 m 求和, 并注意到 $|\alpha|=n-s$ 时, $\sum_{l=0}^{m}(\alpha_l+sq_{il}-n\lambda_l)=0$, 则上式化为

$$\sum_{k,l=0}\lambda_k\lambda_l v_p^{(k)}\left[(D_{\lambda_k}-D_{\lambda_l})S_\gamma^{(i)}(x)+n\sum_{j=1}^{m}\gamma_j(v_j^{(k)}-v_j^{(l)})S_{\gamma-e^j}^{(i)}(x)\right]$$

$$=S_{\gamma+e^p}^{(i)}(x)+S_\gamma^{(i)}(x)\sum_{k,l=0}^{m}(sq_{il}\lambda_k-sq_{ik}\lambda_l)v_p^{(k)},$$

故而

$$S_{\gamma+e^p}^{(i)}(x)=\sum_{k,l=0}^{m}\lambda_k\lambda_l v_p^{(k)}\left[(D_{\lambda_k}-D_{\lambda_l})S_\gamma^{(i)}(x)+n\sum_{j=1}^{m}\gamma_j(v_j^{(k)}-v_j^{(l)})S_{\gamma-e^j}^{(i)}(x)\right]$$
$$+s(v_p^{(i)}-x_p)S_\gamma^{(i)}(x).$$

证毕

由 $S_\gamma^{(i)}(x)$ 的定义经计算易知,

$$|r|=0\text{ 时}, S_0^{(i)}(x)=1. \tag{3}$$

再由引理 1 或引理 2 不难求得,

$$|\gamma|=1\text{ 时}, S_{e^k}^{(i)}(x)=s(v_k^{(i)}-x_k) \tag{4}$$

$$|\gamma|=2\text{ 时}, S_{e^k+e^l}^{(i)}(x)=(n-s)\sum_{j=0}^{m}\lambda_j v_l^{(j)}v_k^{(j)}+s^2(v_l^{(i)}-x_l)(v_k^{(i)}-x_k)$$
$$-(n-s)x_kx_l. \tag{5}$$

由引理 1 和引理 2, 对 $|\gamma|$ 用数学归纳法可得如下推论

推论 $S_\gamma^{(i)}(x)$ 是关于 n 的次数 $\le [\frac{|\gamma|}{2}]$ 的多项式, 是关于 s 的次数 $\le |\gamma|$ 的多项式, 且 $|S_\gamma^{(i)}(x)| \le c(\gamma) s^{|\gamma|} n^{[\frac{|\gamma|}{2}]}$, $c(\gamma) \ge 0$ 是和 γ 有关的常数.

这两个引理合起来, 构成了 $S_\gamma^{(i)}(x)$ 的递推公式. 若记 $S_\gamma(x) = \sum_{i=0}^{m} \lambda_i S_\gamma^{(i)}(x)$, 则 $n^{-|\gamma|} S_\gamma(x)$ 便是算子 $M_{ns}(f;x)$ 的矩量. 当 σ 为二维标准单纯形 Δ 时, 即 $m = 2$, $v^{(0)} = (0,0)$, $v^{(1)} = (1,0)$, $v^{(2)} = (0,1)$. 由(3), (4), (5) 可得[2] 中的引理 2.1.

3　主要结果

定理　$\forall f(x) \in C^{2N}(\sigma)$ (σ 上 $2N$ 阶连续可微函数空间), 则

$$M_{ns}(f;x) - f(x) = \sum_{|\gamma| \le 2N} \frac{1}{\gamma!} D^\gamma f(x) \frac{S_\gamma(x)}{n^{|\gamma|}} + o_s(n^{-N}),$$

其中 $D^\gamma = D^{\gamma_1}\cdots D^{\gamma_m} = \frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1}\cdots\partial x_m^{\gamma_m}}$, $o_s(n^{-N})$ 中的下标 s 指 $n \to \infty$ 时, 无穷小与 s 的取值有关.

证明　由 Taylor 公式,

$$f(x_\alpha^{(i)}) - f(x) = \sum_{\substack{|\gamma| \le 2N \\ |\gamma| \ne 0}} \frac{1}{\gamma!} (x_\alpha^{(i)} - x)^\gamma D^\gamma f(x) + \sum_{|\gamma| = 2N} \frac{1}{\gamma!} (x_\alpha^{(i)} - x)^\gamma [D^\gamma f(\psi_\alpha^{(i)}) - D^\gamma f(x)].$$

这里 $\psi_\alpha^{(i)} = x + \theta(x_\alpha^{(i)} - x)$, $0 < \theta < 1$. 所以

$$M_{ns}^{(i)}(f;x) - f(x) = \sum_{|\alpha| = n-s} (f(x_\alpha^{(i)}) - f(x)) B_\alpha(\lambda) = \sum_{\substack{|\gamma| \le 2N \\ |\gamma| \ne 0}} \frac{1}{\gamma!} \frac{S_\gamma^{(i)}(x)}{n^{|\gamma|}} D^\gamma f(x)$$

$$+ \sum_{|\gamma| = 2N} \frac{1}{\gamma!} \sum_{|\alpha| = n-s} (x_\alpha^{(i)} - x)^\gamma [D^\gamma f(\psi_\alpha^{(i)}) - D^\gamma f(x)] B_\alpha(\lambda).$$

对于 $\delta^{(i)} > 0$, 写

$$\sum_{|\alpha| = n-s} (x_\alpha^{(i)} - x)[D^\gamma f(\psi_\alpha^{(i)}) - D^\gamma f(x)] B_\alpha(\lambda) = \sum_{\|x_\alpha^{(i)} - x\|_c < \delta^{(i)}} \cdots + \sum_{\|x_\alpha^{(i)} - x\|_c \ge \delta^{(i)}} \cdots,$$

这里 $\|x_\alpha^{(i)} - x\|_c = \max_{1 \le j \le m} |(x_\alpha^{(i)} - x)_j|$. $f(x) \in C^{2N}(\sigma) \Rightarrow \forall \epsilon > 0$, $\exists\, \delta^{(i)} > 0$, 当 $\|x_\alpha^{(i)} - x\|_c < \delta^{(i)}$ 时, 有 $|D^\gamma f(\psi_\alpha^{(i)}) - D^\gamma f(x)| < \epsilon/2[c(2\gamma) s^{2|\gamma|}]^{\frac{1}{2}}$. 进而推得

$$\Big| \sum_{\|x_\alpha^{(i)} - x\|_c < \delta^{(i)}} \cdots \Big| \le (\epsilon/2[c(2\gamma) s^{2|\gamma|}]^{\frac{1}{2}}) \sum_{\substack{|\alpha| = n-s \\ \|x_\alpha^{(i)} - x\|_c < \delta^{(i)}}} |x_\alpha^{(i)} - x)^\gamma| B_\alpha(\lambda)$$

$$\le (\epsilon/2[c(2\gamma) s^{2|\gamma|}]^{\frac{1}{2}}) \Big(\sum_{|\alpha| = n-s} (x_\alpha^{(i)} - x)^{2\gamma} B_\alpha(\lambda)\Big)^{\frac{1}{2}} \Big(\sum_{|\alpha| = n-s} B_\alpha(\lambda)\Big)^{\frac{1}{2}}$$

$$\le (\epsilon/2[c(2\gamma) s^{2|\gamma|}]^{\frac{1}{2}}) \Big(\frac{1}{n^{2|\gamma|}} S_{2\gamma}^{(i)}(x)\Big)^{\frac{1}{2}} \le \epsilon/2n^{|\gamma|/2}.$$

记 $M_\gamma = \max_{x \in \sigma} |D^\gamma f(x)|$, 则

$$\Big| \sum_{\|x_\alpha^{(i)} - x\|_c \ge \delta^{(i)}} \cdots \Big| \le 2M_\gamma \sum_{\substack{|\alpha| = n-s \\ \|x_\alpha^{(i)} - x\|_c \ge \delta^{(i)}}} |x_\alpha^{(i)} - x|^\gamma B_\alpha(\lambda)$$

$$\le \frac{2M_\gamma}{\delta^{(i)}} \sum_{\substack{|\alpha| = n-s \\ \|x_\alpha^{(i)} - x\|_c \ge \delta^{(i)}}} \|x_\alpha^{(i)} - x\|_c |x_\alpha^{(i)} - x|^\gamma B_\alpha(\lambda)$$

$$\le \frac{2M_\gamma}{\delta^{(i)}}\Big(\sum_{\substack{|\alpha|=n-s\\ \|x_\alpha^{(i)}-x\|_c\ge\delta^{(i)}}}\|x_\alpha^{(i)}-x\|_c^2 B_\alpha(\lambda)\Big)^{\frac12}\Big(\sum_{\substack{|\alpha|=n-s\\ \|x_\alpha^{(i)}-x\|\ge\delta^{(i)}}}(x_\alpha^{(i)}-x)^{2\gamma}B_\alpha(\lambda)\Big)^{\frac12}$$

$$\le \frac{2M_\gamma}{\delta^{(i)}}\Big(\sum_{j=1}^{m}\frac1n S_{2e^j}^{(i)}(x)\Big)^{\frac12}\Big(\frac{1}{n^{|\gamma|}}S_{2\gamma}^{(i)}(x)\Big)^{\frac12}$$

$$\le \frac{2M_\gamma}{\delta^{(i)}}\Big(\sum_{j=1}^{m}c(2e^j)s^2\frac1n\Big)^{\frac12}c^{\frac12}(2\gamma)s^{|\gamma|}n^{-\frac{|\gamma|}{2}}\le\frac{M(\gamma,m,s)}{\delta^{(i)}}n^{-(\frac{|\gamma|}{2}+\frac12)}.$$

对此 $\delta^{(i)}$ 取足够大的 n, 使 $1/\delta^{(i)}n^{\frac12}<\epsilon/2M(\gamma,m,s)$. 从而 $\Big|\sum_{\|x_\alpha^{(i)}-x\|_c\ge\delta^{(i)}}\cdots\Big|<\epsilon/2n^{|\gamma|/2}$. 所以,

$$M_{ns}^{(i)}(f;x)-f(x)=\sum_{|\gamma|\le 2N}\frac{1}{\gamma!}\frac{S_\gamma^{(i)}(x)}{n^{|\gamma|}}D^\gamma f(x)+o_s(n^{-N}).$$

两边乘 λ_i, 关于 i 从 0 到 m 求和, 即得结论. 证毕

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Asymptotic Formulae of Approximation Error for Smooth Functions by Stancu Operator on Simplex

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Abstract

In this paper, Stancu operators on the generalized simplex in m-dimensional Euclid space are defined and its high order asymptotic formulae of pointwise approximation error for smooth functions is given.

Keywords simplex, stancu operator, $C^{2N}(\sigma)$ space, asympototic expansion.