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Rigidity Theorems of Riemannian Manifold with $\nabla^2 Ric = 0$ *

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Abstract Ricci curvature tensor is denoted by Ric. We study when the manifold which satisfy $\nabla^2 Ric = 0$ become a Einstein manifold or a space form.

Keywords Ricci curvature tensor, Riemann curvature tensor, Einstein space, weyl conformal curvature tensor, scalar curvature.

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1. Introduction

Compact Riemannian manifolds whose Ricci curvature tensors are parallel have been studied in [1], the following two theorems are obtained by the authors of [1]:

Theorem A *Suppose M^n is a compact manifold, $\nabla^2 Ric = 0$, Scalar curvature $R = n(n-1)$. If $2n(n-1) \le \|R_M\|^2 < 2n(n-1) + \left(\frac{n}{3+\sqrt{n-2}}\right)^2$, then M is a space form.*

Theorem B *Suppose M^n is a compact Einstein manifold, scalar curvature $R = n(n-1)$. If $0 \le \|W_M\|^2 < \frac{4}{9}n(n-1)$, then M is a space form.*

R_M and W_M in theorem A, B denote Riemann curvature tensor and Weyl conformal curvature tensor respectively. We will establish similar theorems on manifolds satisfy $\nabla^2 Ric = 0$ and find best constants for theorem A, B. We will also develop some rigidity theorems which observed in [2], [3].

2 Preliminaries

Suppose M is a complete manifold of dimension n. $\{w_1, \cdots, w_n\}$ are local orthonormal frames. We have structure equations:

$$dw_i = -\sum_j w_{ij} \wedge w_j, \tag{1}$$

$$dw_{ij} = -\sum_k w_{ik} \wedge w_{kj} + \Omega_{ij}, \tag{2}$$

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where $\Omega_{ij} = \frac{1}{2}\sum_{k,l} R_{ijkl}w_k \wedge w_l$, R_{ijkl} are Riemann curvature tensors of M. The indexes range from 1 to n.

Ricci curvature are defined by $R_{ij} = \sum_l R_{lilj}$. The covariant derivative are defined by

$$\sum_k R_{ij,k} w_k = dR_{ij} - \sum_m R_{mj} w_{mi} - \sum_m R_{im} w_{mj}, \tag{3}$$

$$\sum_l R_{ij,kl} w_l = dR_{ij,k} - \sum_m R_{mj,k} w_{mi} - \sum_m R_{im,k} w_{mj} - \sum_m R_{ij,m} w_{mk}, \tag{4}$$

$\nabla^2 Ric = 0$ denotes $R_{ij,kl} = 0$. From (3), (4) we have Ricci identity

$$\sum_m R_{mj} R_{mikl} + \sum_m R_{im} R_{mjkl} = 0.$$

If we choose good frames so that $n \times n$ matrix (R_{ij}) is diagonal, then

$$(R_{ii} - R_{jj}) \cdot R_{ijkl} = 0. \tag{5}$$

(5) is very useful.

Lemma 1 *If M is connected, $\nabla^2 Ric = 0$; then the scalar curvature R is a constant.*

Proof **From** (3), (4) **and** $R = \sum_i R_{ii}$, $\nabla^2 Ric = 0$, we have

$$\sum_k R_{,m} w_k = dR, \tag{6}$$

$$dR_{,k} - \sum_m R_{,m} w_{mk} = 0, \tag{7}$$

Taking exterior differential calculus on (7), by (2) we have

$$0 = ddR_{,k} = \sum dR_{,m} \wedge w_{mk} + \sum_m R_{,m} (- \sum_i w_{mi} \wedge w_{ik} + \Omega_{mk}).$$

By (7) we have

$$\sum_m R_{,m} \Omega_{mk} = 0,$$

$$\sum_n R_{,m} R_{mkij} = 0. \tag{8}$$

Using Bianchi identity

$$R_{ijkl,m} + R_{ijlm,k} + R_{ijmk,l} = 0$$

and $\nabla^2 Ric = 0$, by (8) we have

$$\sum_m R_{,m} R_{ijkl,m} = 0.$$

Since $R = \sum_{ij} R_{ijij}$, we have

$$\sum_m R_{,m} R_{,m} = 0.$$

From (6) we have $dR = 0$, R is a constant for connectivity of M. $\square$

Suppose λ is a constant, define

$$D_{ijkl} = R_{ijkl} - \lambda(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$

D_{ijkl} have the following qualities:

$$\|D\|^2 = \sum_{i,j,k,l} D_{ijkl}^2 = \|R_m\|^2 + 2\lambda^2 n(n-1) - 4\lambda R, \tag{9}$$

$$D_{ijkl,m} + D_{ijmk,1} + D_{ijlm,k} = 0, \tag{10}$$

$$\sum_i D_{ijil} = R_{jl} - (n-1)\lambda\delta_{jl}. \tag{11}$$

Letting $i = m$ in (10), by (11) we have

$$\sum_m D_{mjkl,m} + R_{jk,l} - R_{jl,k} = 0.$$

Using $\nabla^2 Ric = 0$ we have

$$\sum_m D_{ijkm,ml} = 0. \tag{12}$$

From (10) we have

$$\sum_m D_{ijkl,mm} = \sum_m R_{ijkm,lm} + \sum_m R_{ijml,km}.$$

Then

$$\begin{aligned}\frac{1}{2}\Delta\|D\|^2 &= \sum_{i,j,k,l,m} D_{ijkl,m}^2 + D_{ijkl}D_{ijkl,mm}\\ &= \|\nabla D\|^2 + 2\sum_{i,j,k,l,m} D_{ijkl}D_{ijkm,lm}\\ &= \|\nabla D\|^2 + 2\sum_{i,j,k,l,m} D_{ijkl}(D_{ijkm,lm} - D_{ijkm,ml})\\ &= \|\nabla D\|^2 + 2\sum_{i,j,k,l,m} D_{ijkl}(D_{hjkm}R_{hilm} + D_{ihkm}R_{hjlm}\\ &\quad + D_{ijhm}R_{hklm} + D_{ijkh}R_{hl}).\end{aligned} \tag{13}$$

Using the method in [1], we have

$$\frac{1}{4}\nabla\|D\|^2 \geq \frac{1}{2}\|\nabla D\|^2 + \frac{n}{2}\lambda\|D\|^2 - \frac{1}{2}(\sqrt{n-2} + 3)\|D\|^3. \tag{14}$$

If M is a Einstein manifold whose Ricci curvature is $(n-1)\lambda$, then

$$\frac{1}{4}\triangle\|D\|^2 \geq \frac{1}{2}\|\nabla D\|^2 + (n-1)\lambda\|D\|^2 - \left(\frac{1}{2} + \frac{n-2}{\sqrt{n(n-1)}}\right)\|D\|^3. \tag{15}$$

One can obtain theorem A, B by (13), (14).

3 Pinching Problems

We will consider the condition for M to become a Einstein manifold. First we have

Theorem 1 *Suppose M is a complete manifold of dimension $n(\geq 3)$, M is connected, $\nabla^2 Ric=0$. Scalar curvature $R \neq 0$. If*

$$\|R_M\|^2 \leq \frac{2R^2}{(n-1)(n-2)}$$

then M is a Einstein manifold or $\|R_M\|^2=\frac{2R^2}{(n-1)(n-2)}$. And $\|R_M\|^2=\frac{2R^2}{(n-1)(n-2)}$ implies that the universal covering space of M is isometry to $M^{n-1}\left(\frac{R^2}{(n-1)(n-2)}\right)\times \mathbf{R}$. Where $M^{n-1}\left(\frac{R^2}{(n-1)(n-2)}\right)$ denotes $(n-1)$-dimensional simply connected space form which sectional curvature is $\frac{R^2}{(n-1)(n-2)}$. $\mathbf{R}$ denotes Euclidean line.

Proof From Lemma 1 we know $R = const$, $\forall p \in M$, choose good frames around p such that (5) is true for p. Then the indexes from 1 to n are divided into s groups:

$$\{1, 2, \cdots, x_1\}, \{x_1+1, x_1+2, \cdots, x_1+x_2\}, \cdots,$$
$$\leq \left\{\sum_{m=1}^{S-1} x_m+1, \sum_{m=1}^{S-1} x_m+2, \cdots, \sum_{m=1}^{S-1} x_m+x_s\right\}, 1 \leq s \leq n.$$

Suppose

$$R_{jj}=a_i, \sum_{m=1}^{i-1} x_m+1 \leq j \leq \sum_{m=1}^{i-1} x_m+x_i, 1 \leq i \leq s,$$

and $a_1, a_2, \cdots, a_s$ are different real numbers. From (5) we have

$$R_{ijkl}=0, \quad \forall k, l=1, 2, \cdots, n \tag{16}$$

if i, j are in different groups.

Clearly,

$$\sum_{i=1}^{s} x_i a_i = R, \tag{17}$$

$$\sum_{i=1}^{s} x_i = n. \tag{18}$$

The square length of Ricci tensor is defined by $\|Ric\|^2=\sum_{i=1}^{s} x_i a_i^2$. If $s=1$, then $x_1=n$, $a_1=\frac{R}{n}$ and $\|Ric\|^2=\frac{R^2}{n}$. If $s \geq 2$, we will prove $\|R_M\|^2=\frac{2R^2}{(n-1)(n-2)}$. In this case $\|Ric\|^2=\frac{R^2}{n-1}$. From continuity of $\|Ric\|^2$ and connectivity of M we know M is a Einstein manifold or $\|R_M\|^2=\frac{2R^2}{(n-1)(n-2)}$. In fact, suppose $i \in \{1, 2, \cdots, x_1\}$, from (16) we have

$$a_1 = R_{ii} = \sum_{l=1}^{x_1} R_{lili}.$$

Then

$$\sum_{l=1}^{x_1} R_{1l1l}^2 \geq \frac{1}{x_1 - 1}\left(\sum_{l=1}^{x_1} R_{1l1l}\right)^2 = \frac{a_1^2}{x_1 - 1}. \tag{19}$$

If $x_1 = 1$, then by (16) we know $a_1 = 0$, define right hand of (19) equal to 0. So

$$\sum_{i,j=1}^{x_1} R_{ijij}^2 \geq \frac{x_1}{x_1 - 1} a_1^2.$$

Similarly, we have

$$\| R_M \|^2 = \sum_{i,j,k,l} R_{ijkl}^2 \geq 2 \sum_{i=1}^{s} \frac{x_i}{x_i - 1} a_i^2. \tag{20}$$

$s \geq 2$ implies $x_i \leq n - 1$, then

$$\| R_M \|^2 \geq \frac{2(n-1)}{n-2} \sum_{i=1}^{s} a_i^2. \tag{21}$$

From (17), (18) and $s \geq 2$, $x_i = 1$ implies $a_i = 0$, using Cauchy inequality we have

$$\sum_{i=1}^{s} a_i^2 \geq \frac{R^2}{(n-1)^2}, \tag{22}$$

and if $\sum_{i=1}^{s} a_i^2 = \frac{R^2}{(n-1)^2}$ then $s = 2$, $\{x_1, x_2\} = \{n-1, 1\}$.

From (21), (22) we know $\| R_M \|^2 \geq \frac{2R^2}{(n-1)(n-2)}$. So from the conditions in theorem we know $\| R_M \|^2 = \frac{2R^2}{(n-1)(n-2)}$. Then we can suppose $x_1 = n-1$, $x_2 = 1$ and $a_1 = \frac{R}{n-1}$, $a_2 = 0$. This implies $\| Ric \|^2 = x_1 a_1^2 + x_2 a_2^2 = \frac{R^2}{n-1}$.

Now we suppose $\| R_M \|^2 = \frac{2R^2}{(n-1)(n-2)}$. From (19), (20) we have

$$-R_{\{ijji\}} = R_{ijij} = \frac{R}{(n-1)(n-2)},\quad 1 \leq i, j \leq n-1,\quad i \neq j,$$

other R_{ijkl} equal to 0. From (13), if $\lambda = 0$, computation gives

$$\| \nabla R_M \|^2 = 0,$$

then M is locally symmetric. By De Rham's decomposition theorem in [4], we know the universal covering space of M is isometry to $M^{n-1}\frac{R}{(n-\ )(n-2)} \times \mathbf{R}$. □

Corollary 1 *Under the condition of theorem 1, if* $\| R_M \|^2 < \frac{2R^2}{(n-1)(n-2)}$, *then M must be a Einstein manifold. Specially, if $n \geq 11$, $R > 0$, M is compact, then M must be a space form.*

Proof The first part is from theorem 1. If $n \geq 11$, then

$$\frac{2R^2}{(n-1)(n-2)} \leq \frac{2R^2}{n(n-1)} + \left(\frac{R}{n}\right)^2 \left[\frac{1}{2} + \frac{n-2}{\sqrt{n(n-1)}}\right]^{-2},$$

from (9), (15), where $\lambda = \frac{R}{n(n-1)}$. We know that if

$$\| R_M \|^2 < \frac{2R^2}{n(n-1)} + \left(\frac{R}{n}\right)^2 \left[\frac{1}{2} + \frac{n-2}{\sqrt{n(n-1)}}\right]^{-2},$$

then M is a space form. Now the second part is clear. □

Remark Corollary 1 is better than theorem A. If $n \geq 11$, the pinching constant $\frac{2R^2}{(n-1)(n-2)}$ is the best one.

Next we will consider the case that $n = 3$ or 4.

Theorem 2 $n = 3$, $\nabla^2 Ric = 0$. *M is a complete connected manifold, then M is a space form or the universal covering space of M is isometry to $M^2(\frac{R}{2}) \times \mathbf{R}$.* Where R is the scalar curvature of M.

Proof From Lemma 1 we have $R = $ const. If $R = 0$, from (16), (17), (18) we know $Ric = 0$. Then M is flat because Weyl conformal curvature tensor of three-manifold is vanishing. If $R \neq 0$, from (16), (17), (18) we know (i) $s = 1$, $x_1 = n$, $a_1 = \frac{R}{n}$ or (ii) $s = 2$, $\{x_1, x_2\} = \{n-1, 1\}$, $\{a_1, a_2\} = \{\frac{R}{n-1}, 0\}$. From the proof of theorem 1 we know M is a Einstein manifold or the universal covering space is isometry to $M^2(\frac{R}{2}) \times \mathbf{R}$. One knows that three dimensional Einstein manifold must be a space form. □

Theorem 3 $n = 4$, $\nabla^2 Ric = 0$, *M is a complete connected manifold. If M is compact, then M is a Einstein manifold or the universal covering space is isometry to $M^3(\frac{R}{6}) \times \mathbf{R}$ or $M^2 \times N^2$. where M^2 and N^2 are manifolds whose Gauss curvature are constants, and they are simply connected.*

Proof From (16), (17), (18) we know

(i) $s = 1$, $R_{11} = R_{22} = R_{33} = R_{44} = \frac{R}{4}$;

(ii) $s = 2$, $R_{11} = R_{22} = a$, $R_{33} = R_{44} = b$, $a \neq b$, $a + b = \frac{R}{2}$;

(iii) $s = 2$, $R_{11} = R_{22} = R_{33} = \frac{R}{3}$, $R_{44} = 0$.

From $\nabla^2 Ric = 0$ we know

$$\frac{1}{2}\Delta \| Ric \|^2 = \| \nabla Ric \|^2$$

then $\| Ric \|^2 = $ const because M is compact.

In case (ii), from (16) we know $R_{1212} = R_{2121} = -R_{1221} = -R_{2112} = a$, $R_{3434} = R_{4343} = -R_{3443} = -R_{4334} = b$, other R_{ijkl} equal to 0. In case (iii), from (16) we know $R_{ijij} = -R_{ijji} = \frac{R}{6}$, $1 \leq i, j \leq 3$, $i \neq j$, other R_{ijkl} equal to 0. From (13), $\lambda = 0$, we have

$$\frac{1}{2}\Delta \| R_M \|^2 = \| \nabla R_M \|^2 \tag{23}$$

for both (ii) and (iii).

If (i) happens for some point $p \in M$, then M is a Einstein manifold because $\| Ric \|^2$ is the smallest in the three cases. Now we suppose (ii) or (iii) is true, then from (23) we know $\nabla R_M = 0$ because M is compact. So M is locally symmetric.

If (ii) happens for some point $p\in M$, from $\|Ric\|^2=2(a^2+b^2)=\text{const}$, $a+b=\frac{R}{2}=$ const we know $a=$ const $b=$ const, $\|R_M\|^2=4(a+b)^2=\text{const}\geq\frac{R^2}{2}$. If (iii) happens for some point $p\in M$, then $\|R_M\|^2=\frac{R^2}{3}<\frac{R^2}{2}$. So (ii) happens for every point of M or (iii) happens for every point of M because of the connectivity of M. By De Rham's decomposition theorem in [4] we know that the universal covering space is isometry to $M^2\times N^2$ in the case (ii) or $M^3(\frac{R}{6})\times\mathbf{R}$ in the case (iii). □

Another condition also can make M become a Einstein manifold. Suppose $p\in M$, V denotes $(r+1)$-dimensional linear space of T_pM. If $\forall V$ and orthogonal unit vectors $\{e_1,\cdots,e_r\}\subset V$, the following

$$\sum_{i=1}^{r}R_m(v,e_i,v,e_i)>0$$

is true for every unit vector $v\in V$ then we say $Ric(r)>0$ at $p\in M$. Similarly we can define $Ric(r)<0$.

Theorem 4 *M is a complete Riemannian manifold of dimension n, M is connected, $\nabla^2 Ric=0$. If $Ric(r)>0$ or $Ric(r)<0$ for every point of M, where $r=[\frac{n+1}{2}]$, r denotes the integral part of $\frac{n+1}{2}$. Then M is a Einstein manifold. Specially, if the sectional curvature $K>0$ or $K<0$ for every point of M, then M is a Einstein manifold.*

Proof Suppose M is not a Einstein manifold, then $\exists\, p\in M$, such that $R_{11}\neq R_{nn}$. We can suppose $R_{11}=R_{22}=\cdots=R_{ii}$, $R_{jj}=\cdots=R_{nn}$, $1\leq i<j\leq n$. From (5) we have

$$R_{1k1k}=0,\quad i+1\leq k\leq n,\tag{24}$$

$$R_{nlnl}=0,\quad 1\leq l\leq j-1.$$

Since $(n-1)+(j-1)=(n-1)+(j-i)\geq n$, we can suppose $n-i\geq r=[\frac{n+1}{2}]$. Then from (24) we have

$$\sum_{k=n-r+1}^{n}R_{1k1k}=0,$$

which contradicts with the condition in the theorem.

If $K>0$ or $K<0$, clearly $Ric(r)>0$ or $Ric(r)<0$, the theorem is proved. □

4 Global Rigidity}

We will establish some rigidity theorems similar to [2], [3] for connected manifold which satisfy $\nabla^2 Ric=0$. If M is compact, then V_M denotes the volume of M and d_M denotes the diameter of M. We have Sobolev inequality holds in [5].

$$\|f\|_{\frac{2n}{n-2}}\leq c(n)\cdot v_M^{-\frac{1}{n}}\cdot[d_M\cdot\|\nabla f\|_2+\|f\|_2],\ \forall f\in c^\infty(M),\tag{25}$$

where $c(n)$ is a constant depending only on n. Define $\sigma=\|D\|$, where $\lambda=\frac{R}{n(n-1)}$.

Theorem 5 *Suppose M is a compact manifold dimension $n(\geq 4)$, $\nabla^2 Ric = 0$. Scalar curvature $R = n(n-1)$. If*

$$\|\sigma\|\Big|_{\frac{n}{2}} < c_1(n)\cdot v_M^{\frac{2}{n}},\ c_1(n) = \min\{2\sqrt{\frac{n(n-1)}{n-2}}, \frac{4(n-2)}{3\pi^2 n^2\cdot c^2(n)}\},$$

then M is a space form.

Proof From the proof of theorem 3 we know $\|Ric\|^2 = $ const, then from the proof of theorem 1 we know that if M isn't a Einstein manifold then $\|R_M\|^2 \geq 2\cdot\frac{n-1}{n-2}\cdot n^2$. From (9) we have

$$\sigma \geq \sqrt{2\cdot\frac{n-1}{n-2}n^2 - 2n(n-1)} = 2\sqrt{\frac{n(n-1)}{n-2}}.$$

So M is a Einstein manifold under the condition of the theorem.

From (15) we have

$$\Delta\sigma + 3\sigma^2 - 2(n-1)\sigma \geq 0. \tag{26}$$

Multiply (26) by $\sigma^{\frac{n-2}{2}}$. Integration by parts gives

$$3\int_M \sigma\cdot\sigma^{\frac{n}{2}} \geq 2(n-1)\cdot\int_M \sigma^{\frac{n}{2}} + \frac{n-2}{2}\int_M \sigma^{\frac{n}{2}-2}(\nabla\sigma)^2$$

$$= 2(n-1)\cdot\int_M \sigma^{\frac{n}{2}} + \frac{8(n-2)}{n^2}\int_M |\nabla\sigma^{\frac{n}{4}}|^2.$$

Applying Hölder's inequality, $\int_M \sigma\cdot\sigma^{\frac{n}{2}} \leq (\int_M \sigma^{\frac{n}{2}})^{\frac{2}{n}}\cdot(\int_M \sigma^{\frac{n}{2}\cdot\frac{n}{n-2}})^{\frac{n-2}{n}}$, from (25) we have

$$6c^2(n)(\int_M \sigma^{\frac{n}{2}})^{\frac{2}{n}}\cdot v_M^{-\frac{2}{n}}\cdot(d_M^2\cdot\int_M |\nabla\sigma^{\frac{n}{4}}|^2 + \int_M \sigma^{\frac{n}{2}})$$

$$\geq 2(n-1)\int_M \sigma^{\frac{n}{2}} + \frac{8(n-2)}{n^2}\int_M |\nabla\sigma^{\frac{n}{4}}|^2.$$

Applying Myer's theorem, $d_M \leq \pi$. Then if $(\int_M \sigma^{\frac{n}{2}})^{\frac{2}{n}}\cdot v_M^{-\frac{2}{n}} \leq \frac{4(n-2)}{3\pi^2 n^2 c^2(n)}$, then $\int_M \sigma^{\frac{n}{2}} = 0$, $\sigma \equiv 0$, M is a space form. □

Theorem 6 *Suppose M is a compact manifold of dimension $n(\geq 4)$, $\nabla^2 Ric = 0$, scalar curvature $R = -n(n-1)$. Then give $D > 0$, $\exists\,\epsilon = \epsilon(n, D)$, such that if $d_M \leq D$ and $\int_M \sigma^{\frac{n}{2}} \leq v_M\cdot\epsilon$ then $\sigma \equiv 0$, M is a space form.*

Proof Suppose $\epsilon < [2\sqrt{\frac{n(n-1)}{n-2}}]^{\frac{n}{2}}$, from the proof of theorem 5, we know M is a Einstein manifold. Now from theorem 4 in [3], the conclusion is clear. □

Theorem 7 *Suppose M is an open manifold of dimension $n(\geq 10)$, $\nabla^2 Ric = 0$, scalar curvature $R = -n(n-1)$, if $\sigma \leq \frac{(n-1)(n-9)}{(n-1)^2+12}\cdot(1-\frac{1}{9}e^2)$ and for some point $p \in M$, $\lim_{r\to+\infty} e^{-\delta_n\cdot r}\int_{B(p,r)}\sigma^2 = 0$, where $\delta_n = \frac{1}{3}\sqrt{(n-1)(n-9)}$ and $B(p, r)$ denotes the geodesic ball of radius r around p, then $\sigma \equiv 0$, M is a space form.*

Proof Take $\epsilon = \frac{(n-1)(n-9)}{(n-1)^2+12} \cdot (1-\frac{1}{9}e^2)$. Then if $\sigma \leq \epsilon$, the sectional curvature K satisfies

$$K \leq -(1-\epsilon) < 0.$$

From theorem 4 we know M is a Einstein manifold. From (15) we have

$$\Delta\sigma + 3\sigma^2 + 2(n-1)\sigma \geq 0.$$

So

$$\Delta\sigma + (3\epsilon + 2(n-1))\sigma \geq 0. \tag{27}$$

Multiply (27) by $\sigma\eta^2$, where η is a cut off function with compact support in M. Integration by parts gives

$$(3\epsilon + 2(n-1))\int_M (\sigma\eta)^2 \geq \int_M |\nabla(\sigma\eta)|^2 - \int_M |\nabla\eta|^2\sigma^2. \tag{28}$$

By [6] we have

$$\int_M |\nabla(\sigma\eta)|^2\sigma^2 \geq \frac{1}{4}(n-1)^2(1-\epsilon)\int_M (\sigma\eta)^2.$$

By (28) we have

$$\int_M |\nabla\eta|^2\sigma^2 \geq [\frac{1}{4}(n-1)(n-9) - (\frac{1}{4}(n-1)^2+3)\cdot\epsilon]\cdot\int_M(\sigma\eta)^2$$

$$= \frac{1}{4}e^2\delta^2(n)\int_M(\sigma\eta)^2.$$

Choosing $\eta(x) = \eta(d(p,x))$, where $d(p,x)$ denotes distant function.

$$\eta(t) = \begin{cases} 1, & t \leq r, \\ \frac{R-t}{R-r}, & r \leq t \leq R,. \\ 0, & t \geq R. \end{cases}$$

We obtained by (29)

$$\frac{1}{(R-r)^2}\int_{(p,R)}\sigma^2 \geq \frac{1}{4}e^2\sigma^2(n)\int_{(p,r)}\sigma^2. \tag{30}$$

For any $r_0 > 0$, take $r_j = 2\sigma^{-1}(n)\cdot j + r_0$, $j \geq 0$, it then follows from (30) that

$$\int_{(p,r_j)}\sigma^2 \geq e^2\int_{(p,r_{j-1})}\sigma^2 \geq e^{2j}\int_{(p,r_0)}\sigma^2 = e^{\sigma(n)(r_j-r_0)}\int_{(p,r_0)}\sigma^2,$$

$$\int_{B(p,r_0)\}}\sigma^2 \leq e^{-\sigma(n)(r_j-r_0)}\cdot\int_{(p,r_j)}\sigma^2.$$

Letting $r_j \to +\infty$, One obtains $\sigma \equiv 0$ on $B(p, r_0)$. It's easy to see $\sigma \equiv 0$ on M. i.e., M is a space form. $\square$

The authors don't know the case $R = 0$. Perhaps there is no general rigidity theorems in this case.

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$\nabla^2 Ric = 0$ 的 Riemann 流形的刚性定理

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摘　要

本文用 Ric 表示里奇曲率张量，研究了 $\nabla^2 Ric = 0$ 的黎曼流形什么时候成为爱因斯坦流形或空间形式