Journal of Mathematical Research & Exposition
Vol. 18, No. 1, 11-16, February 1998

# Continuous Spectrum Measure of a Class of Schrödinger Operators*

*Shang Pengjian*
(Dept. of Appl. Math., Northern Jiaotong University, Beijing 100044)

**Abstract** We consider the discrete Schrödinger operator acting on $l^2(\mathbf{Z})$ with the potential $V_n$, the sequence consisting of $k+1$ symbols: $\{0, 1, 2, \cdots, k\}$, and prove that it exhibits purely continuous spectrum.

**Keywords** discrete, Schrödinger operator, potential, continuous spectrum.
**Classification** AMS(1991) 47H12/CCL O177.6

## 1. Introduction

We study one-dimensional discrete Schrödinger operator acting on $l^2(\mathbf{Z})$:

$$(HU)_n = U_{n+1} + U_{n-1} + \lambda V_n U_n \tag{1}$$

where $V_n$ is a sequence consisting of $k+1$ symbols: $\{0, 1, \cdots, k\}$. Various such one dimensional models have been investigated[1,2,3], namely with random or almost periodic potential. Also, much works have been devoted to studying the Schrödinger operator with a potential generated by the Fibonacci sequence. The discovery of quasi-crystals gives physical grounds to study of such kind of potential which can be thought as one-dimensional model[4,5,6]. In [1], Kohmoto, Kadanoff and Tang proposed a model where $V_n$ is given by

$$V_n = \chi_A(n\omega + \theta) \tag{2}$$

where $\chi_A$ is the characteristic function of an interval $A$ on the circle and the argument of $\chi_A$ has to be understood modulo 1. Where $\omega$ was the golden mean, $A$ was the interval $(-\omega^3, \omega^2]$ and $\theta$ was 0. These authors conjectured that the spectral measure is singular continuous. In this paper, we show that the operator (1) with the potential $V_n$, consisting of $k+1$ symbols: $\{0, 1, 2, \cdots, k\}$, exhibits purely continuous spectrum.

## 2 Notation and Result

### 2.1 On continued fraction expansions

* Received July 19, 1994.

Since we will make an extensive use of the continued fraction expansion (CFE) of the irrational number, we must first recall some basic notation. Any irrational number $\omega > 0$ can be written in a unique way as

$$\omega = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} = [a_0; a_1, a_2, \cdots], \tag{3}$$

where the integers $a_i$'s are called the partial quotients of CFE. If we truncate this expansion, we obtain a sequence of rational approximants

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_i}}} = \frac{p_i}{q_i} \tag{4}$$

called the principal convergents of the CFE. Their numerators and denominators both obey linear recursive relations as

$$p_i = a_i p_{i-1} + p_{i-2}, \quad q_i = a_i q_{i-1} + q_{i-2}. \tag{5}$$

In addition, we have for $i \geq 0$

$$p_i q_{i+1} - p_{i+1} q_{i+1} = 1. \tag{6}$$

2.2 Sequences generated by a circle map

Let us first define characteristic function with $\Delta$ a given number between 0 and 1:

$$\chi_\Delta(x) = \mathrm{Int}(x) - \mathrm{Int}(x - \Delta) = \begin{cases} 1 & \text{if } 0 \leq \mathrm{Frac}(x) < \Delta, \\ 0 & \text{if } \Delta \leq \mathrm{Frac}(x) < 1. \end{cases} \tag{7}$$

Here Int $(x)$ and Frac $(x) = x - \mathrm{Int}(x)$ denote integer and fractional parts of $x$, respectively. It is easily seen that $\chi_\Delta(x)$ is a 1-periodic function. Consider sequence $\{\chi_\Delta(n\omega + \theta)\}$, consisting of two symbols 0 and 1. This sequence may also be generated by a circle map, namely unit circumference, as shown by Fig. 1

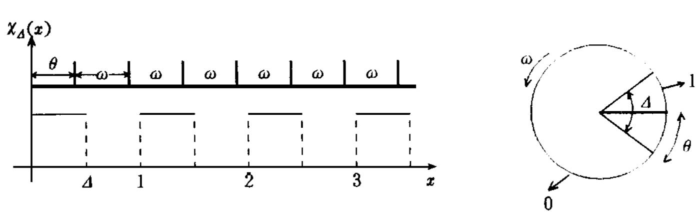


Fig 1

Let us define an intermediate labelling sequence on the circle (unit circumference). A sequence $L_n$ is associated with the sequence of numbers $\alpha_n = \mathrm{Frac}(\theta + n\omega)$ by the rule

$$L_n = \begin{cases} k, & \text{if } 0 \le \alpha_n < x_1, \\ k-1, & \text{if } x_1 \le \alpha_n < x_2, \\ \vdots & \vdots \\ 1, & \text{if } x_{k-1} \le \alpha_n < x_k, \\ 0, & \text{if } x_k \le \alpha_n < 1, \end{cases} \tag{8}$$

where $0 < x_1 < x_2 < \cdots < x_k < 1$. This sequence consists of $k+1$ symbols: 0, 1, 2, $\cdots$, $k$. And it may also be generated by a circle map (namely unit circumference), as shown by Fig. 2. The sequence $\{\chi_\Delta(\theta + n\omega)\}$ is recovered by choosing $k = 1, x_1 = \Delta$.

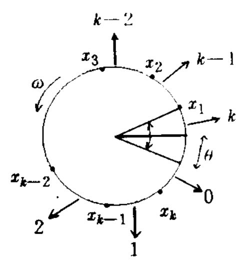


Fig. 2

2.3 Results

We study discrete Schrödinger operators (1). In the following theorems $V_n$ is the sequence $L_n$ in (8), $\omega$ an irrational number. We use the notation $H(\theta, \lambda)$ to take into account the dependence of $H$ on $\lambda$ and $\theta$.

**Theorem** *For Lebesque almost every $\omega$ and for any $x_1, x_2, \cdots, x_k$ $(0 < x_1 < x_2 < \cdots < x_k)$, then for Lebesque almost every the spectral measure of $H(\theta, \lambda)$ is continuous for any value of $\lambda$.*

## 3. Lemmas and Proof of Theorem

**Lemma 1** *Let $U$ be a solution of the eigenvalue equation $HU = EU$ and suppose that there exists an integer $r$ such that $V_{n+ir} = V_n$ for $i = -1, 1$ and $0 < n \le r$ (Hypothesis $H_1$), then we have for all $E$:*

$$\max(\|Y_{-r}\|, \|Y_r\|, \|Y_{2r}\|) \ge \|Y_0\|/2, \tag{9}$$

*where $Y_n$ is the vector $(U_n, U_{n+1})^T$.*

**Proof** For any solution $U$ of $HU = EU$ one can write

$$U_{n+1} + U_{n-1} + \lambda V_n U_n = EU_n.$$

By the use of transfer matrices, we have

$$Y_n = T(n) Y_{n-1}, \tag{10}$$

where

$$Y_n = \begin{bmatrix} U_n \\ U_{n+1} \end{bmatrix}, \quad T(n) = \begin{bmatrix} 0 & 1 \\ -1 & E - \lambda V_n \end{bmatrix}. \tag{11}$$

Let

$$M(n) = T(n)T(n-1)\cdots T(1), \tag{12}$$

then

$$Y_n = M(n)Y_0. \tag{13}$$

Hypothesis $H_1$ implies that

$$\left.\begin{aligned} Y_r &= M(r)Y_0 \\ Y_0 &= M(r)Y_{-r} \\ Y_{2r} &= M(r)Y_r \end{aligned}\right\}. \tag{14}$$

Since the determinant of $M(r)$ is 1, the matrix $M(r)$ satisfies:

$$M^2(r) - \mathrm{tr}(M(r)) \cdot M(r) + I = 0, \tag{15}$$

where tr(•) is the trace of matrix (•), $I$ is $2 \times 2$ unit matrix.

i) The case $|\mathrm{tr}(M(r))| \leq 1$.

Apply the characteristic equation (15) to any vector $x \in \mathbf{C}^2$, and take the norm, this yields

$$\| x \| = \| t_r(M(r))M(r)x - M^2(x)x \| \leq \|M(r)x\| + \|M^2(r)x\|.$$

Let $x = Y_0 = \begin{bmatrix} U_0 \\ U_1 \end{bmatrix}$, and consider equalities (14), one gets

$$\| Y_0 \| \leq 2\mathrm{Max}\{ \| Y_r \|, \| Y_{2r} \| \}. \tag{16}$$

ii) The case $|t_r(M(r))| > 1$.

The equation (15) implies that

$$M(r) = \frac{1}{\mathrm{tr}(M(r))}M^2(r) + \frac{1}{\mathrm{tr}(M(r))}I.$$

For any $x \in \mathbf{C}^2$, we have

$$M(r)x = \frac{1}{\mathrm{tr}(M(r))}M^2(r)x + \frac{1}{\mathrm{tr}(M(r))}x.$$

Take the norm

$$\|M(r)x\| = \left\| \frac{1}{\mathrm{tr}(M(r))}M^2(r)x + \frac{1}{t_r(M(r))}x \right\| \leq \|M^2(r)x\| + \|x\|.$$

Let $x = M^{-1}(r)Y_0$ and consider equalities (14), one gets

$$\| Y_0 \| \leq 2\mathrm{Max}\{ \| Y_r \|, \| Y_{-r} \| \}. \tag{17}$$

Thus Lemma 1 is a consequence of (16) and (17).

**Lemma 2** *For almost every $\omega$ there exists for almost every $\theta$ a sequence $r_n(\theta)$ for which the Hypothesis $H_1$ of Lemma 1 is fulfilled.*

**Proof** Let $\frac{p_n}{q_n}$ be the $n^{\text{th}}$ principal convergent of the CFE of the irrational number $\omega$ so that one gets

$$q_{n+1} = a_n q_n + q_{n-1}, \quad p_{n+1} = a_n p_n + p_{n-1},$$

where $a_n$ is the $n^{\text{th}}$ partial quotient of CFE of $\omega$. The rate of convergence of the $\frac{p_n}{q_n}$ to $\omega$ is given by

$$\left|\omega - \frac{p_n}{q_n}\right| \leq \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| = \frac{1}{q_n q_{n+1}} \leq \frac{1}{a_n q_n^2}. \tag{18}$$

Now the set $E(n)$ of value of $\theta$ such that Hypothesis $H_1$ is fulfilled with $r = q_n$ is given by

$$E(n) = E_1(n) \cap E_2(n) \cap \cdots \cap E_k(n) = \bigcap_{i=1}^{k} E_i(n),$$
$$E_i(n) = \{\theta \mid \inf_{0<m\leq q_n} (|\mathrm{Frac}(m\omega + \theta) - x_i|) > q_n \left|\omega - \frac{p_n}{q_n}\right|\}, \tag{19}$$

where $x_i (i = 1, 2, 3, \cdots, k)$ are the endpoints in (8), see Fig. 2. Indeed, (19) expresses the fact that Frac $(m\omega + \theta)$ for $m$ in $[1, q_n]$ must be distant from $x_i$ by at least the phase shife corresponding to a translation of $\pm q_n$ in order to ensure the exact repetition of the potential considered in Lemma 1. Clearly we have

$$\mu(E_i(n)) \geq 1 - 2q_n^2 \left|\omega - \frac{p_n}{q_n}\right|, \tag{20}$$

where $\mu$ is the Lebesgue measure on $[0, 1]$, thus using (18):

$$\mu(E(n)) \geq 1 - 2(k+1) q_n^2 \left|\omega - \frac{p_n}{q_n}\right| \geq 1 - \frac{2(k+1)}{a_n}. \tag{21}$$

Now, it is known [7] that for a. e. $\omega$ one gets:

$$\limsup_{n\to\infty} a_n = +\infty, \tag{22}$$

which yields that

$$\limsup_{n\to\infty} \mu(E(n)) = 1,$$

whence

$$\mu(\limsup_{n\to\infty} E(n)) = 1.$$

Consequently, for a. e. $\omega$, there exists for $\mu$ a. e. $\theta$ an infinite sequence $r_k(\theta) = q_{n_k}(\theta)$ such that $H_1$ is fulfilled.

**Proof of Theorem** It suffices to show that any solution $U$ of the eigenvalue equation $HU = EU$ cannot decay at infinity.

From Lemma 2 we know that there exists, for a. e. $\omega$ and for a. e. $\theta$, an infinite increasing sequence $r_n$ satisfying the Hypothesis $H_1$ of Lemma 1. In this case $\max(\|Y_{-r_n}\|, \|Y_{r_n}\|, \|Y_{2r_n}\|)$ does not tend to $o$ as $n$ tend to $+\infty$, therefore $H(\theta, \lambda)$ cannot have eigenvectors in $l^2(\mathbf{Z})$ for a. e. $\theta$ and all $\lambda$.

## References


[1] M. Kohmoto, L. Kadanof, C. Tang, *Localization problem in one dimension: Mapping and escape*, Phys. Rev. Lett., **50**(1983), 1870.

[2] M. Kohmoto, Y. Oono, *Cantor spectrum for an almost periodic schrödinger equation and a dynamical map*, Phys. Lett., **145**(1984), 120A.

[3] S. Ostlund, R. Pandit, D. Rang, H. J. Schellnhuber, E. D. Siggia, *One-dimensional schrödinger equation with an almost periodic potential*, Phys. Rev. Lett., **50**(1983), 1873.

[4] A. Ya. Gordon, Usp. Math. Nauk 31(1976), 257.

[5] J. Avron, B. Simon, Bull. Am. Math. Soc. 6(1982), 81.

[6] B. Simon, Adv. Appl. Math., **3**(1982), 463.

[7] A. Ya. Khinχn, Continued fractions, The University of Chicago Press, 1964, 60.


# 一类 Schrödinger 算子的连续谱测度


**商　朋　见**

(北方交通大学应用数学系, 北京 100044)


## 摘　　要


**考虑 $l^2(\mathbf{Z})$ 上的离散 Schrödinger 算子, 其势 $V_n$ 是一个由 $k+1$ 个符号 $\{0, 1, 2, \cdots, k\}$ 构成的序列. 证明了它具有纯连续谱.**