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The Total Chromatic Number of Graphs with an Unique Major Vertex of Degree Four°

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Abstract The total chromatic number $\chi_T(G)$ of a graph G is the least number k such that G admits a total coloring with k colors. In this paper, it is proved that $\chi_T(G) = \Delta(G) + 1$ for all graphs with an unique major vertex of degree 4.

Keywords graph, total coloring, major vertex.
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1. Introduction

All graphs in this paper are finite and simple. Undefined signs and concepts can be found in [1].

Given a graph G, $N_G(v)$, $d_G(v)$ and $\Delta(G)$ denote the neighbour set of a vertex v in G, the degree of v in G and the maximum degree of vertices of G, respectively. For any two elements u and v in $V(G) \cup E(G)$, we say that u and v cover each other if u and v are adjacent or incident, and say that u and v are independent to each other otherwise. A subset S of $V(G) \cup E(G)$ is called an independent set of G if all elements of S are mutually independent. A subset M of $E(G)$ is called a perfect matching of G if M is an independent set of G and covers all vertices of G. We say a vertex v of a graph G is a k- *vertex* in G if $d_G(v) = k$ and say v is a major vertex of G if $d_G(v) = \Delta(G)$.

For an edge $uv \in E(G)$ and a vertex $w \notin V(G)$, graph $H = G - uv + uw + vw$ is called an edge-subdivision of G. If two graphs G_1 and G_2 can be constructed from a same graph G by a serials of edge-subdivision, then we say that G_1 and G_2 are homemorphic to each other.

A k-total-coloring of a graph G is an assignment of k colors to $V(G) \cup E(G)$ such that no adjacent elements or incident elements receive the same color. Let σ be a k-total coloring of a graph G and v be a vertex of G, we use $C_\sigma(v)$ to denotes the set of colors assigned to v or the edges incident with v in σ. The total chromatic number $\chi_T(G)$ of a graph G is the least number k such that G

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admits a k-total coloring. A famous conjecture named total coloring conjecture[2] claims that $\Delta(G)+1 \leq \chi_T(G) \leq \Delta(G)+2$ for any graph G.

In [3], Zhang Zhongfu and Wang Jianfang proposed a new conjecture about total coloring.

Conjecture 1 $\chi_T(G)=\Delta(G)+1$ *for all graphs with an unique major vertex.*

Given a graph G, if we add a new edge to $E(G)$ by joining a vertex $v \notin V(G)$ to a major vertex of G, then the new graph has an unique major vertex. So, this conjecture 1 implies total coloring conjecture.

It is very easy to verify that Conjecture 1 holds for graphs of maximum degree 3 and bipartite graphs. In this paper, it is proved that Conjecture 1 also holds for graphs of maximum degree 4. Following lemma is needed in the proof.

Lemma 1.1[1] *Every* **2**-*connected* **3**-*regular graph has a perfect matching.*

2 Results and Proofs

Let G be a graph of maximum degree 3. A 4-total coloring of G is called perfect if all the 3-vertices of G receive a same color, color α say, and each of the 2-vertices and 1-vertices of G receive a color different from α. We refer to a perfect 4-total coloring as 4-PTC.

(First, let us show some usful lemmas.)

Lemma 2.1 *Let G be a* **2**-*connected* **3**-*regular graph and H be a graph obtained by subdividing each edge of G once a time. Then, H admits a* **4**-*PTC.*

Proof By the defination of H,

$$V(H) = V(G) \cup E(G),$$

$$E(H) = \{uv \mid u \in V(G),\ v \in E(G),\ u \text{ and } v \text{ are incident in } G\}$$

and each 2-vertex of H corresponds to an unique edge of G.

By Lemma 1.1, G has a perfect matching M. Let V_m denote the set of 2-vertices of H which correspond to the edges of M. Since $G-M$ consists of cycles $C_1, C_2, \cdots, C_l$, then, $H-V_m$ consists of even cycles $C'_1, C'_2, \cdots, C'_l$. Let M_i be a perfect matching in C'_i, $i=1,2,\cdots,l$. Then

$$S_1 = \bigcup_{i=1}^{l} M_i \cup V_m$$

and

$$S_2 = \{v \in V(H): d_H(v) = 3\}$$

are two independent sets of H.

Since each component B of $H-\bigcup_{i=1}^{l} M_i$ is isomorphic to a path $P = v_1v_2v_3v_4v_5$ of length 4 and the vertex of B which corresponds to v_3 in P is a 3-vertex in H, so, the set of edges of $H-\bigcup_{i=1}^{l} M_i$ can be partitioned into two independent sets of H, S_3' and S_4' say. Because all the 2-vertices of H are independent to each other and for any 2-vertex v of $H-V_m$, exactly one of S_3'; S_4' uncovers v, so, all the elements of $S_3' \cup S_4' \cup$ {2-vertex of $V(H)-V_m$} can be partitioned into two independent sets, S_1 and S_2 say. Clearly, if we color the elements of S_i with color i for each $i \in \{1, 2, 3, 4\}$, then, this is a 4-PTC of H. □

Lemma 2.2 *Let G be a 2-connected graph of maximum degree 3 and all major vertices of G are independent to each other. Then, G admits a 4-PTC.*

Proof By induction on the number of 2-vertices of G. Let

$$V^2(G) = \{v \in V(G) : d_G(v) = 2\}.$$

If there is a subset S of $V(G)$ such that $G[S]$ is a cycle and there are only two 3-vertices u and w of G in S, then $\chi_T(G) = 4$ while G contains exactly two 3-vertices and it is not difficult to verify that $\chi_T(G) = \chi_T(G-E(G[S]) + uw)$ while G has at least three 3- vertices

Without loss of generality, suppose that there is not such a subset S in $V(G)$, i.e, there are at least three 3-vertices on each induced cycle of G. Then, G must be homemorphic to a 2-connected 3-regular graph H and $|V^2(G)| \geq |E(H)| = l$.

By Lemma 2.1, $\chi_T(G) = 4$ while $|V^2(G)| = l$. Suppose that G admits a 4-PTC while $|V^2(G)| = k \geq l$.

Let G be a 2-connected graph of maximum degree 3, all 3-vertices of G are mutually independent and $|V^2(G)| = k + 1 > l$. Given a 2-vertex v of G such that $N_G(v) = \{u_1, u_2\}$, $d_G(u_1) = 3$ and $N_G(u_2) = \{v, u_3\}$, let $G_1 = G-v + u_1u_2$. Then by the induction hypothesis, G_1 admits a 4-PTC σ_0 such that all the 3-vertices of G receive a same color, color 1 say, and each 2-vertex of G receives a color different from color 1.

While $\sigma_0(u_3) = 1$, let $\sigma(u_1v) = \sigma_0(u_1u_2)$, $\sigma(vu_2) = 1$, $\sigma(u_2) = i \in \{2, 3, 4\}-\{\sigma_0(u_2u_3)\}$ and $\sigma(v) = j \in \{2, 3, 4\}-\{i, \sigma(vu_1)\}$.

If $\sigma_0(u_3) \neq 1$, then u_3 is a 2-vertex of G, suppposе $N_G(u_3) = \{u_2, u_4\}$.

Figure 1:

While $\sigma_0(u_3) = \sigma_0(u_1u_2) = i$ (see figure 1(a)), let $\sigma(u_1v) = \sigma_0(u_1u_2)$, $\sigma(vu_2) = 1$, $\sigma(u_2) = j \in \{3, 4\}-\{i\}$ if $\sigma_0(u_3u_4) = 1$ and $\sigma(u_2) = \sigma_0(u_3u_4)$ if $\sigma_0(u_3u_4) \neq 1$, $\sigma(v) = j \in \{2, 3, 4\}-\{i, \sigma(u_2)\}$ and $\sigma(u_2u_3) = j \in \{2, 3, 4\}-\{i, \sigma_0(u_3u_4)\}$.

While $\sigma_0(u_3) = j \neq \sigma_0(u_1u_2) = i$ and $\sigma_0(u_2u_3) = 1$ (see figure 1(b)). Let $\sigma(u_1v) = \sigma_0(u_1u_2)$, $\sigma(vu_2) = 1$, $\sigma(u_2) = h \in \{3, 4\}-\{i\}$ if $\sigma_0(u_3u_4) = 1$ and $\sigma(u_2) = \sigma_0(u_3u_4)$ if $\sigma_0(u_3u_4) \neq 1$, let $\sigma(v) = k \in \{2, 3, 4\}-\{i, \sigma(u_2)\}$ and $\sigma(u_2u_3) = h \in \{2, 3, 4\}-\{i, \sigma_0(u_3u_4)\}$.

While $\sigma_0(u_3) = j \neq \sigma_0(u_1u_2) = i$ and $\sigma_0(u_2u_3) \neq 1$. Let $\sigma(u_1v) = \sigma_0(u_1u_2)$, $\sigma(vu_2) = 1$, $\sigma(v)$

$= \sigma_0(u_3)$, $\sigma(u_2) = k \in \{2, 3, 4\} - \{\sigma_0(u_2u_3), \sigma_0(u_3)\}$.

Other elements not mentioned above receive the same color as in σ_0. Clearly, σ is a 4-PTC. This complete the proof. □

Lemma 2.3 *Let G be a seperable graph of maximum degree 3 and all 3-vertices of G are independent to each other. Then, G adm its a 4-PTC.*

Proof Without loss of generality, suppose that $\delta(G) = 2$. It is very easy to prove this lemma by induction on the number of cutvertex of degree 2 in G. □

Now, we have the main theorem of this paper.

Theorem 2.4 *Let G be a graph with an unique major vertex of degree 4. Then, $\chi_T(G) = 5$.*

Proof Let G be a graph with an unique major vertex u of degree 4. In case that G does not contains matching M which covers u and each 3-vertex of $N(u)$, we can verify that $\chi_T(G) = 5$ directly. Without loss of generality, suppose that M is a maximum matching of G which covers u and each 3-vertex of $N(u)$, and let $H = G - M$. Then $\Delta(H) = 3$ and all major vertices of H are mutually independent. By Lemma 2.3 and Lemma 2.4, H admits a 4-PTC σ_0 with color set $C = \{1, 2, 3, 4\}$ and all 3-vertices receive color 1. It is clearly that for each 2-vertex v of H, the elements adjacent to v or incident with v receive at most 3 colors in σ_0. We shall construct a 5-total coloring σ of G from σ_0.

First, let $\sigma(w) = \sigma_0(w)$ for each $w \in V(H) \cup E(H)$, and let $\sigma(xy) = 5$ for each edge xy of M having $\sigma_0(x) \neq \sigma_0(y)$.

If each edge of M has received a color, then σ is a 5-total coloring of G, otherwise, let $e = xy$ be an edge of M and $\sigma_0(x) = \sigma_0(y) = l$. Let $N_H(x) = \{x_1, x_2\}$ and $N_H(y) = \{y_1, y_2\}$. Without loss of generality, suppose that $l = 4$ and both $\{x_1, x_2, xx_1, xx_2\}$ and $\{y_1, y_2, yy_1, yy_2\}$ receive exactly 3 colors

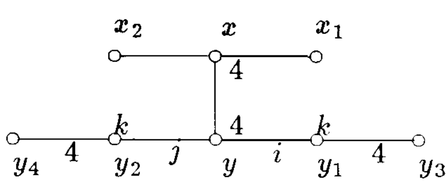

Figure 2:

Case 1 $\sigma_0(y_1) = \sigma_0(y_2) = k \neq 4$, $\sigma_0(yy_1) = i \neq 4$ *and* $\sigma_0(yy_2) = j \neq 4$.

In this case, if both y_1 and y_2 are 3-vertices of H, then at least one of them, y_1 say, is not u, let $\sigma(y_1) = 5$, $\sigma(yy_1) = k$, $\sigma(y) = i$ and $\sigma(xy) = 5$; if y_1 is a 2-vertex in H and $4 \notin C_{\sigma_0}(y_1)$, then let $\sigma(yy_1) = 4$, $\sigma(y) = i$ and $\sigma(xy) = 5$; if y_1 is a 2-vertex in G, we can modify σ_0 such that y_1 and yy_2 receive a same color, color j say, and y and yy_1 receive color i and k respectively, then we have $\sigma_0(x) \neq \sigma_0(y)$. So we assume that both y_1 and y_2 are 2-vertices in H, $4 \in C_{\sigma_0}(y_1) \cap C_{\sigma_0}(y_2)$ (See figure 2) and $y_iz_i \in M$ ($i = 1, 2$).

If one of y_3 and y_4, y_4 say, is a 3-vertex in H, let $\sigma(y_2) \in \{i, j\} - \{\sigma_0(z_2)\}$, $\sigma(y_2y) = k$,

$\sigma(y) \in \{i, j\}-\{\sigma(y_2)\}$, $\sigma(yy_1) = \sigma(y_2)$ and $\sigma(xy) = 5$. Therefore, we assume that both y_4 and y_3 are 2-vertices in H (See figure 3, where the slanted edges y_1z_1 and y_2z_2 are edges of M).

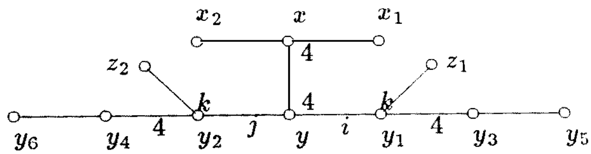

Figure 3:

According to the set of colors assigned to z_1, y_3, y_3y_5, z_2, y_4, and y_4y_6 in σ_0, we can reassign colors to y_4y_2, y_2, y_2y, y, yy_1, y_1y_3 and y_3 such that y receives a color different from color 4. for example, if $\sigma_0(z_1) = j$, $\sigma_0(y_3) = j$ and $\sigma_0(y_3y_5) = i$, then let $\sigma(y_1) = i$, $\sigma(yy_1) = k$, $\sigma(y) = j$, $\sigma(yy_2) = i$ and $\sigma(xy) = 5$.

Case 2 $\sigma_0(y_1) = \sigma_0(yy_2) = j$ *and* $\sigma_0(x_1) = \sigma_0(xx_2) = i$.

If one of x_1 and y_1, say y_1, is 2-vertex in H (see figure 4), then let $\sigma(yy_1) \in \{4, \sigma_0(y_2)\}-\{\sigma_0(y_1y_3)\}$, $\sigma(y) = \sigma_0(yy_1)$ and $\sigma(xy) = 5$.

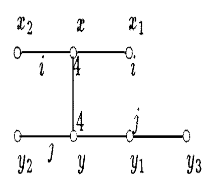

Figure 4:

Case 2 1 *One of x_1 and y_1, y_1 say, is a 3-vertex in H which is not u (see figure 5(a)). In this case, if one of x_1 or x_2, x_1 say, is also a 3-vertex in H which is not u, then $M' = M - xy + xx_1 + yy_1$ is a matching of G which covers u and each 3-vertex in $N_G(u)$ and $|M'| > |M|$, a contradiction. So, we can assume that both $\sigma_0(x_1) \neq 5$ and $\sigma_0(x_2) \neq 5$ hold, let $\sigma(x) = 5$, $\sigma(xy) = 4$, $\sigma(y) = k$, $\sigma(yy_1) = 5$, and let $H = H - yy_1 + xy$, $M = M - xy + yy_1$.*

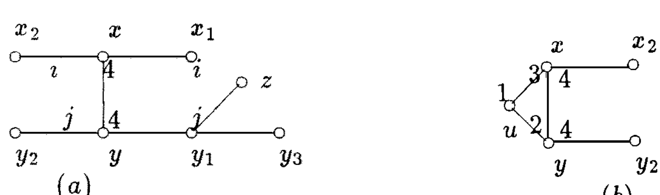

Figure 5:

Case 2 2 *$y_1 = x_1 = u$. Without loss of generality, suppose that $\sigma_0(ux) = 3$ and $\sigma_0(uy) = 2$ (see figure 5(b)). According to the set of colors assigned to yy_2, y_2, xx_2 and x_2, it is very easy to reas-*

sign the colors to x, y, yy_2, y_2, xx_2 *and* x_2 *such that* x *and* y *receive different colors. For example, while* $\sigma_0(xx_2) = 2$ *and* $\sigma_0(x_2) = 3$, *we can reassign colors as following: Let* $\sigma(xx_2) \in \{1, 4\} - \{C_{\sigma_0}(x_2)\}$, $\sigma(x) = 2$ *and* $\sigma(xy) = 5$.

Let σ_0 be the restriction of σ to H, repeat the above process till each edge of M receive a color. σ is a 5-total coloring of G. □

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具有唯一4度最大度点的图的全色数

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摘　要

一个图G的全色数$\chi_T(G)$是使得$V(G) \cup E(G)$中相邻或相关联元素均染不同颜色的最少颜色数. 文中证明了, 若图G只有唯一的一个4度最大度点, 则$\chi_T(G) = \Delta(G) + 1$.