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Quasi-Fast Completeness and Inductive Limits of Webbed Spaces*

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Abstract Let $(E, \xi) = \text{indlim}\,(E_n, \xi_n)$ be an inductive limit of locally convex spaces. We say that (DST) holds if each bounded set in (E, ξ) is contained and bounded in some (E_n, ξ_n). We introduce a property which is weaker than fast completeness, quasi-fast completeness, and prove that for inductive limits of strictly webbed spaces, quasi-fast completeness implies that (DST). By using De Wilde's theory on webbed spaces, we also give some other conditions for (DST). These results improve relevant earlier results.

Keywords locally convex spaces, inductive limits, webbed spaces
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As well known, inductive limits of locally convex spaces appeared in the theory of topological vector spaces for the first time when some of the common spaces of distribution theory were topologized in the natural way. For example, if Ω is an open subset of R^n, then a distribution on Ω is defined to be a continuous linear functional on $D(\Omega)$. Here $D(\Omega)$ denotes the set of all infinitely differentiable functions $f: \Omega \to R$ such that f has compact support in Ω and the natural topology on $D(\Omega)$ is the inductive limit topology defined by the system $\{D(K_n)\}$, where $\{K_n\}$ is a sequence of compact subsets such that K_n is contained in the interior of K_{n+1} and $\Omega = \bigcup_{n=1}^{\infty} K_n$, and $D(K_n) = \{f \in D(\Omega): \text{supp.}\, f \subseteq K_n\}$ has the topology defined by the seminorms $p_{n,m}(f) = \sup\{f^{(k)}(x): |k| \le m, x \in K_n\}, m = 1, 2, 3, \cdots$. Indeed, $D(\Omega)$ with the above topology, i.e., $D(\Omega) = \text{indlim}\, D(K_n)$, is a strict inductive limit of Fréchet spaces. The main positive results on countable strict inductive limits are due to Dieudonné, Schwartz and Köthe (see [1], p. 58). Let $E_1 \subseteq E_2 \subseteq \cdots$ be a sequence of locally convex, Hausdorff, topological vector spaces (abbreviated l.c.s.) and $(E, \xi) = \text{indlim}\,(E_n, \xi_n)$ be their locally convex, Hausdorff, inductive limit with respect to the continuous identity maps $id: (E_n, \xi_n) \to (E_{n+1}, \xi_{n+1})$. For brevity, denote by (DST) each set B bounded in (E, ξ) is contained and bounded in some (E_n, ξ_n) (See [6]).

Dieudonné Schwartz Theorem ([1], p. 59) states that a set $B \subseteq E$ is ξ-bounded if and only if it is contained and bounded in some (E_n, ξ_n), i.e., (DST) holds, provided that $\xi_{n+1}|E_n = \xi_n$

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and E_n is closed in (E_{n+1}, ζ_{n+1}) for every $n \in N$. However some non-strict inductive limits also have many applications and it is possible to extend Dieudonné Schwartz Theorem to general inductive limits (see [5]-[15]). In this paper, we shall investigate inductive limits of strictly webbed spaces. Strictly webbed spaces have many important properties and include Fréchet spaces, strong duals of metrizable l. c. s., sequentially complete (DF)-spaces, etc. (see [3] and [16]). Hence the theory of inductive limits of strictly webbed spaces has wide applications. We shall introduce the definition of quasi-fast completeness, which is weaker than fast completeness. By using DeWilde's theory on webbed spaces, we prove that quasi-fast completeness implies (DST) for inductive limits of strictly webbed spaces. We also give some other conditions for (DST). These results improve related earlier results.

First we establish the following Lemma 1, which is indeed a variety of the localization theorem for strictly webbed spaces.

Lemma 1 *Let $(E, \zeta) = \text{indlim}\,(E_n, \zeta_n)$ be an inductive limit of strictly webbed spaces and (F, η) be a Fréchet space. If a linear map $t: (F, \eta) \to (E, \zeta)$ is sequentially closed and B is a bounded subset of (F, η), then there exists $n \in N$ such that $t(B)$ is a contained and bounded in (E_n, ζ_n).*

Proof For each $p \in N$, let $\{C^{(p)}_{n_1}, \cdots, n_k\}$ be the strict web on (E_p, ζ_p), then $\{D_{n_1}, \cdots, n_k\} = \{C^{(n_1)}_{n_2}, \cdots, n_k\}$. is the strict web on (E, ζ), where D_{n_1} is defined to be E_{n_1}. (see [3], II, §35, 4). By the localization theorem for strictly webbed spaces (see [3], II, §35, 6(1)), there exist a sequence $n_1, n_2, \cdots$ of natural numbers and a sequence $\alpha_1, \alpha_2, \cdots$ of positive numbers such that $t(B) \subseteq \alpha_k D_{n_1, \cdots, n_k}$ for every $k \in N$. That is, $t(B) \subseteq \alpha_k C^{(n_1)}_{n_2}, \cdots, n_k \subseteq E_{n_1}$ for every $k \in N$. Let U be any neighborhood of o in (E_{n_1}, ζ_{n_1}) and $\{\rho_{k-1}\}$ be the sequence of positive numbers corresponding to $\{C^{(n_1)}_{n_2, \cdots, n_k}\}$ (see [3], II, §35, 1), then there exists $k_0 \in N$ such that $\rho_{k-1} C^{(n_1)}_{n_2, \cdots, n_k} \subseteq U$ for any $k \geq k_0$. Thus

$$t(B) \subseteq \alpha_k C^{(n_1)}_{n_2, \cdots, n_k} \subseteq \frac{\alpha_k}{\rho_{k-1}} (\rho_{k-1} C^{(n_1)}_{n_2, \cdots, n_k}) \subseteq \frac{\alpha_k}{\rho_{k-1}} U$$

for any $k \geq k_0$. Hence $t(B)$ is contained and bounded in (E_{n_1}, ζ_{n_1}). This completes the proof.

We recall that an absolutely convex subset B of l. c. s. (E, ζ) is calleda Banach disk if span $[B]$ with the gauge of B is a Banach space and a l. c. s. (E, ζ) is called fast complete if each bounded set in (E, ζ) is contained in some bounded Banach disk in (E, ζ) (see [16]). In the sequel, a topology ζ on E weaker than a topology ζ on E means that ζ is strictly weaker than or equal to ζ. And a locally convex, Hausdorff, linear topology is shortly denoted by a l. c. topology. Now let's introduce a property on l. c. s. which is weaker than fast completeness. A l. c. s. (E, ζ) is called to be quasi-fast complete if for each bounded set B in (E, ζ), there exists a $l.\,c$ topology ζ on E weaker than ζ and a bounded Banach disk C in (E, ζ) such that $B \subseteq C$. It's easy to see that (E, ζ) is quasi-fast complete if and only if for each bounded set B in (E, ζ) there exists a dense subspace H of $(E', \sigma(E', E))$ such that B is contained in some bounded Banach disk of $(E, \sigma(E,$

H)). Obviously, fast completeness implies quasi- fast completeness, but the converse is not true. In fact, if there exists a l. c. topology ζ on E such that (E, ξ) is fast complete and ζ is weaker than ζ, then (E, ξ) is quasi- fast complete. Hence there are a lot of locally convex spaces which are quasi- fast complete but aren't fast complete. For example, let $(X, \|\cdot\|)$ be an infinite dimensional Banach space. Then there exists a linear functional f on X which is not continuous. For any $x \in X$, define $p(x) = \|x\| + |f(x)|$, then (X, p) is an incomplete normed space and the topology generated by p is strictly stronger than one generated by $\|\cdot\|$. Obviously, (X, p) is quasi- fast complete, but it isn't fast complete, since the normed space (X, p) is fast complete if and only if it is complete. In [9], we proved that for inductive limits of strictly webbed spaces fast completeness implies (DST). Now we give the following improvement:

Theorem 1 *Let $(E, \xi) = \operatorname{indlim}(E_n, \zeta_n)$ be an inductive limit of strictly webbed spaces. If (E, ξ) is quasi- fast complete, then* (DST) *holds.*

Proof Let B be any bouunded set in (E, ξ), then there exists a bounded Banach disk C in (E, ξ) such that $B \subseteq C$, where ζ is some l. c. topology weaker than ζ. Suppose that (F, η) is the span $[C]$ with the gauge of C, then (F, η) is a Banach space and B is bounded in (F, η). Obviously, the identity map i: $(F, \eta) \to (E, \xi)$ is continuous and hence i: $(F, \eta) \to (E, \xi)$ is closed. Thus i: $(F, \eta) \to (E, \xi)$ is closed. Using Lemma 1, we know that $B = i(B)$ is contained and bounded in some (E_n, ζ_n).

Theorem 2 *Let $(E, \xi) = \operatorname{indlim}(E_n, \zeta_n)$ be an inductive limit of strictly webbed spaces. If for any $n \in N$, there exists an absolutely convex neighborhood U_n of o in (E_n, ζ_n) and there exists a l. c. topology ζ_n on E comparable with ζ such that U_n is relatively ζ_n- compact, then* (DST) *holds.* }

Proof Let $\overline{U}_n^{\zeta_n}$ and $\overline{U}_n^{\zeta}$ denote respectively the closure of U_n in (E, ζ_n) and in (E, ξ).

(I) Suppose that ζ_n is weaker than ζ, then $\overline{U}_n^{\zeta} \subseteq \overline{U}_n^{\zeta_n}$. By the hypothesis, $\overline{U}_n^{\zeta_n}$ is ζ_n- compact. Let (F, η) be the span $[\overline{U}^{\zeta_n}_n]$ with the gauge of $\overline{U}_n^{\zeta_n}$, then (F, η) is a Banach space. Obviously the identity map i: $(F, \eta) \to (E, \zeta_n)$ is continuous and hence i: $(F, \eta) \to (E, \xi)$ is closed. Using Lemma 1, we know $\overline{U}_n^{\zeta_n} = i(\overline{U}_n^{\zeta_n})$ is contained and bounded in some (E_m, ζ_m). Since $\overline{U}_n^{\zeta} \subseteq \overline{U}_n^{\zeta_n}$, we conclude that $\overline{U}_n^{\zeta}$ is contained and bounded in some (E_m, ζ_m).

(II) Suppose that ζ_n is stronger than ζ. By the hypothesis, $\overline{U}_n^{\zeta_n}$ is ζ_n- compact, then $\overline{U}_n^{\zeta_n}$ is ζ - compact and $\overline{U}_n^{\zeta_n} = \overline{U}_n^{\zeta}$ is ζ- compact. As before, let (F, η) be the span $[\overline{U}_n^{\zeta_n}]$ with the gauge of $\overline{U}_n^{\zeta_n}$, then (F, η) is a Banach space and the identity map i: $(F, \eta) \to (E, \xi)$ is continuous, and certainly is closed. Using Lemma 1, we know that $\overline{U}_n^{\zeta_n} = \overline{U}_n^{\zeta}$ is contained and bounded in some (E_m, ζ_m).

Combining (I) with (II), we always have: $\overline{U}_n^{\zeta}$ is contained and bounded in some (E_m, ζ_m). By Theorem 2 in [9], we conclude that (DST) holds.

Remark 1 If each identity map i_n: $(E_n, \zeta_n) \to (E_{n+1}, \zeta_{n+1})$ is weakly compact (or compact), then there exists an absolutely convex neighborhood U_n of o in (E_n, ζ_n) such that $\overline{U}_n^{E_{n+1}}$ is $\sigma(E_{n+1}, E'_{n+1})$

- compact (or ζ_{g+1} - compact). We remark that it 's all the same to $\overline{U_n}^{E_{n+1}}$ whehter the closure is taken in (E_{n+1}, ζ_{g+1}) or in $(E_{n+1}, \sigma(E_{n+1}, E'_{n+1}))$, since U_n is convex. Now the identity map i_{n+1}: $(E_{n+1}, \sigma(E_{n+1}, E'_{n+1})) \to (E, \sigma(E, E'))$ (or i_{n+1}: $(E_{n+1}, \zeta_{g+1}) \to (E, \zeta)$) is continuous, hence $\overline{U_n}^{E_{n+1}}$ is $\sigma(E, E')$ - compact (or ζ- compact). Namely U_n is relatively $\sigma(E, E')$ - compact (or relatively ζ- compact). Thus the condition in Theorem **2** is satisfied. Hence Theorem **2** can be regarded as a generalization of the relevant results on weakly compact and compact inductive limits (see [**17**]).

Theorem 3 *Let $(E, \zeta) = \text{indlim}\,(E_n, \zeta_g)$ be an inductive limit of strictly webbed spaces. If there exists a sequence $U_1 \subseteq U_2 \subseteq \cdots$ of absolutely convex o- neighborhoods U_n in (E_n, ζ_g) such that $\overline{U_n}^{\zeta_n}$ is ζ_g- sequentially complete, where ζ_g is a l. c. topology on E weaker than ζ, then* (DST) *holds.*

Proof Let B be any bounded set in (E, ζ). Without loss of generality, we may assume that B is absolutely convex. From the proof of Theorem 1 in [8], we can see that B is contained in $n\overline{U_n}^{\zeta}$ for some $n \in N$, and hence B is contained in $n\overline{U_n}^{\zeta_n}$. Obviously $\overline{B}^{\zeta_n} \subseteq n\overline{U_n}^{\zeta_n}$ and $\overline{B}^{\zeta_n}$ is a sequentially complete, bounded disk in (E, ζ_g). let (F, η) be the span $[\overline{B}^{\zeta_n}]$ with the gauge of $\overline{B}^{\zeta_n}$, then (F, η) is a Banach space. Since the identity map i: $(F, \eta) \to (E, \zeta_g)$ is continuous, i: $(F, \eta) \to (E, \zeta)$ is closed. Using Lemma 1, we know that $\overline{B}^{\zeta_n}$ is contained and bounded in some (E_m, ζ_m) and so B is contained and bounded in some (E_m, ζ_m).

Remark 2 In Theorem 3, if all topologies ζ_g are all ζ, then we obtain the following corollary: If there exists a sequence $U_1 \subseteq U_2 \subseteq \cdots$ of absolutely convex o- neighborhoods U_n in (E_n, ζ_g) such that $\overline{U_n}^{\zeta}$ is ζ- sequentially complete, then (DST) holds. Particularly, if all (E_n, ζ_g) are Fréchet spaces and there exists a sequence $U_1 \subseteq U_2 \subseteq \cdots$ of absolutely convex o - neighborhoods U_n in (E_n, ζ_g) such that $\overline{U_n}^{\zeta}$ is ζ- sequentially complete, then (DST) holds. This is just Theorem 2 in [12]. Hence Theorem 3 is a generalization of Theorem 2 in [12].

Repeating the argument analogous to the first half of the proof of Theorem 3 in [12], we can obtain the following more general results:

Theorem 4 *Let $(E, \zeta) = \text{indlim}\,(E_n, \zeta_g)$ be an inductive limit of strictly webbed spaces. If there exists a sequence $U_1 \subseteq U_2 \subseteq \cdots$ of absolutely convex o- neighborhoods U_n in (E_n, ζ_g) such that for any ζ- null sequence $\{x_i: i \in N\}$ contained in $\overline{U_n}^{\zeta}$, $\sum_{i=1}^{\infty} c_i x_i$ is convergent in (E, ζ), then* (DST) *holds. Here, ζ is any l. c. topology on E weaker than ζ and $c = (c_i) \in l^1$ such that $\|c\|_1 \leq 1$.*

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拟速完备性与网状空间的诱导极限

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摘 要

设 $(E,\xi)=\operatorname{indlim}(E_n,\xi_n)$ 为局部凸空间的诱导极限. 称(DST)成立若 (E,ξ) 中每个有界集含于且有界于某 (E_n,ξ_n). 引进一种弱于速完备性的性质, 叫做拟速完备性, 并证明了对于严格网状空间的诱导极限, 拟速完备性蕴涵(DST). 利用 DeWilde 的理论, 也给出了关于(DST)的其他条件. 这些结果改进了已有的有关结论.